STABILITY OF FRACTIONAL ORDER TIME DELAY SYSTEMS

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Abstract. In this paper, some basic results of the stability criteria of fractional order system with time delay as well as free delay are presented. Also, they are obtained and presented sufficient conditions for finite time stability for (non)linear (non)homogeneous as well as perturbed fractional order time delay systems. Several stability criteria for this class of fractional order systems are proposed using a recently suggested generalized Gronwall inequality as well as "*classical*" Bellman-Gronwall inequality. Some conclusions for stability are similar to that of classical integer-order differential equations. Last, a numerical example is given to illustrate the validity of the proposed procedure.

1. Introduction

The question of stability is of main interest in control theory. Also, the problem of investigation of time delay system has been exploited over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.,[1]. Delays are inherent in many physical and engineering systems. In particular, pure delays are often used to ideally represent the effects of transmission, transportation, and inertial phenomena. This is because these systems have only limited time to receive information and react accordingly. Such a system cannot be described by purely differential equations, but has to be treated with differential difference equations or the so called differential equations with difference variables. Delay differential equations (DDEs) constitute basic mathematical models for real phenomena, for instance in engineering, mechanics, and economics, [2]. The basic theory concerning the stability of systems described by equations of this type was developed by Pontryagin in 1942. Also,important works have been written by Bellman and Cooke in 1963, [3]. The presence of time delays in a feedback control system leads to a closed-loop characteristic equation which involves the exponential type transcendental terms. The exponential transcendentality brings infinitely many isolated roots, and hence it makes the stability analysis of time-delay systems a challenging task. It is well recognized that there is no simple and universally applicable practical algebraic criterion, like the Routh–Hurwitz criterion for stability of delay-free systems, for assessing the stability of linear time-invariant time-delayed (LTI-TD) systems. On the other side, the existence of pure time delay, regardless if it present in the control or/and state, may cause undesirable system transient response, or generally, even an instability. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov`s second method, or on using idea of matrix measure,[4-7]. The analysis of time-delay systems can be classified such that the stability or stabilization criteria involve the delay element or not. In other words, delay independent criteria guarantee global asymptotic stability for any time-delay that may change from zero to infinity. As there is no upper limit to time-delay, often delay independent results can be regarded as conservative in practice, where unbounded time-delays are not so realistic. In practice one is not only interested in system stability (e.g. in the sense of Lyapunov), but also in bounds of system trajectories. A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined *a priori* in a given problem. Besides that, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval. These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. Thus, the analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concern. Motivated by "brief discussion" on practical stability in the monograph of LaSalle and Lefschet,[8] and Weiss and Infante,[9] have introduced various notations of stability over finite time interval for continuous-time systems and constant set trajectory bounds. A more general type of stability ("practical stability with settling time", practical exponential stability, etc.) which includes many previous definitions of finite stability was introduced and considered by Grujić,[10,11]. Concept of finite-time stability, called "final stability", was introduced by Lashirer and Story, [12] and further development of these results was due to Lam and Weiss,[13]. Recently, finite-time control/stabilization, and methods for stability evaluation of linear systems on finite time horizont are proposed by Amato *et al.,* [14,15], respectively. Also, analysis of linear time-delay systems in the context of finite and practical stability was introduced and considered in [16-18] and as well as finite-time stability and stabilization [19].

Recently there have been some advances in control theory of fractional (non-integer order) dynamical systems for stability questions such as robust stability, bounded input–bounded output stability, internal stability, finite time stability, practical stability, root-locus, robust controllability, robust observability, etc. For example, regarding linear fractional differential systems of finite dimensions in state-space form, both internal and external stabilities are investigated by Matignon,[20].Some properties and (robust) stability results for linear, continuous, (uncertain) fractional order state-space systems are presented and discussed [20,21].However, we can not directly use an algebraic tools as for example Routh-Hurwitz criteria for the fractional order system because we do not have a characteristic polynomial but pseudopolynomial with rational power-multivalued function. An analytical approach was suggested by Chen and Moore,[22], who considered the analytical stability bound using Lambert function W. Further, analysis and stabilization of fractional (exponential) delay systems of retarded/neutral type are considered [23,24], and BIBO stability [25]. Whereas Lyapunov methods have been developed for stability analysis and control law synthesis of integer linear systems and have been extended to stability of fractional systems, only few studies deal with non-Lyapunov stability of fractional systems. Recently, for the first time, finite-time stability analysis of fractional time delay systems is presented and reported on papers [26,27]. Here, a Bellman-Gronwall`s approach is proposed, using "*classical"* Bellman-Gronwall inequality as well as a recently obtained generalized Gronwall inequality reported in [28] as a starting point. The problem of sufficient conditions that enable system trajectories to stay within the a priori given sets for the particular class of (non)linear (non)autonomous fractional order time-delay systems has been examined.

2. Fundamentals of fractional calculus

Fractional calculus (FC) as an extension of ordinary calculus has a 300 years old history. FC was initiated by Leibniz and L`Hospital as a result of a correspondence which lasted several months in 1695. Both Leibniz and L`Hospital, aware of ordinary calculus, raised the question of a noninteger differentiation (order $n = 1/2$) for simple functions. Subsequent mention of fractional derivatives was made, in some context or the other by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1919, etc. [29]. In that way, the theory of fractional-order derivative was developed mainly in the 19th century. Since from 19th century as a foundation of fractional geometry and fractional dynamics, the theory of FO, in particular, the theory of FC and FDEs and researches of application have been developed rapidly in the world. The modern epoch started in 1974 when a consistent formalism of the fractional calculus has been developed by Oldham and Spanier,[4], and later Podlubny,[6]. Applications of FC are very wide nowadays, in rheology, viscoelasticity, acoustics, optics, chemical physics, robotics, control theory of dynamical systems, electrical engineering, bioengineering and so on, [4-12]. In fact, real world processes generally or most likely are fractional order systems. The main reason for the success of applications FC is that these new fractional-order models are more accurate than integer-order models, i.e. there are more degrees of freedom in the fractional order model. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a ”memory” term in a model. This memory term insure the history and its impact to the present and future. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy transmission line [17] or diffusion of the heat through a semi-infinite solid, where heat flow is equal to the half-derivative of the temperature [6].In his 700 pages long book on Calculus, 1819 Lacroix [30] developed the formula for the n-th derivative of $y = x^m$, m – is a positive integer, $D^n x^m = \frac{m!}{(m-n)!} x^{m-n}$ where $n\ (\leq m)$ is an integer. Replacing the factorial symbol by the Gamma function, he further obtained the formula for the fractional derivative

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} \tag{1}$$

where α and β are fractional numbers and Gamma function $\Gamma(z)$ is defined for $z > 0$ by the so-called *Euler integral of the second kind*:

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx,\quad \Gamma(z+1) = z\Gamma(z) \tag{2}$$

On the other hand, Liouville (1809-1882) formally extended the formula for the derivative of integral order n

$$D^n e^{ax} = a^n e^{ax} \Rightarrow D^\alpha e^{ax} = a^\alpha e^{ax},\ \alpha - arbitrary\, order \quad (3)$$

Using the series expansion of a function, he derived the formula known as *Liouville`s first formula for fractional derivative,* where α may be rational, irrational or complex.

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x} \quad (4)$$

where $f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x),\ \mathrm{Re}\, a_n > 0$. However, it can be only used for functions of the previous form. Also, it was J. B. J. Fourier,[31] who derived the functional representation of function

$$f(t) = \frac{1}{2\pi}\int_R\int_R f(\zeta)\cos\left(\xi(x-\zeta)\right) d\zeta d\xi\ , \quad (5)$$

where he also formally introduced the fractional derivative version. In 1823, Abel considered a mechanical problem, namely Abel's mechanical problem [32]. In the absence of friction, the problem is reduced to an integral equation

$$\int_0^y (y-z)^{-1/2} u(z)dz = \sqrt{2g} f(y),\quad y \in [0, H], \quad (6)$$

where $u(z) = \sqrt{1+\phi'^2(z)}, \phi(z)$ is an increasing function, g is the constant downward acceleration, $f(y)$ is a prescribed function. Then Abel solved (6) in [33]. Also an Abel transform of a sufficiently well behaved function u was generalized to

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} u(t)dt,\quad a < x < b\ , \quad (7)$$

where $-\infty \le a < b \le \infty,\ \alpha \in (0,1)$ and $\Gamma(.)$ is the well known Euler's gamma function. Here, it is assumed the solution of classical Abel integral equation exists and the fractional derivative with order $\alpha \in (0,1)$ exists in $L^1(a,b)$, [34], so we have following results:

Lemma1. Consider, for $\alpha \in (0,1),\ -\infty \le a < b \le \infty$, the classical Abel integral equation

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} u(t)dt = f(x),\quad a < x < b\ , \quad (8)$$

Then there exists at most one solution of equation (8) in $L^1(a,b)$. Moreover, if the function f is absolutely continuous on [a, b], then equation (8) has a solution in $L^1(a,b)$, given by (9)

$$u(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha} f(t)dt = f(x), \quad a<x<b\,, \tag{9}$$

If a and $f(a)$ are finite, then

$$u(x)=\frac{1}{\Gamma(1-\alpha)}\left(f(a)(x-a)^{-\alpha}+\int_a^x (x-t)^{-\alpha} f'(t)dt, \right), \quad a<x<b\,, \tag{10}$$

If a is finite and f is extended by 0 to the left of a, then

$$u(x)=\frac{1}{\Gamma(1-\alpha)}\left(\int_{a-0}^x (x-t)^{-\alpha} df(t) \right), \quad a<x<b\,, \tag{11}$$

If $a=-\infty$ is finite and $\lim_{x\to-\infty}|x|^{1-\alpha} f(x)=0$ then

$$u(x)=\frac{1}{\Gamma(1-\alpha)}\left(\int_{-\infty}^x (x-t)^{-\alpha} df(t) \right), \quad -\infty<x<b\,, \tag{12}$$

From the viewpoint of fractional calculus, we can see that (9)–(12) are just some other forms of fractional derivatives, with order $\alpha\in(0,1)$, under some different hypotheses on f. Fractional derivatives are typically treated as a particular case of pseudo-differential operators. Since they are nonlocal and have weakly singular kernels, the study of fractional differential equations seems to be more difficult and less theories have been established than for classical differential equations. In 1832-1837 a series of papers by Liouville [35,36] reported the earliest form of the fractional integral, though not quite rigorously from the mathematical point of view. The formula was taken as follows

$$D^{-p}\phi(x)=\frac{1}{(-1)^p\Gamma(p)}\int_0^\infty \phi(x+t)t^{p-1}dt, \quad -\infty<x<\infty,\ p>0\,, \tag{13}$$

That is now called the Liouville form of fractional integral with the factor $(-1)^p$ being omitted. Next the significant work was done by Riemann [37], who wrote that paper in 1847 when he was just a student. But it was published until 1876, ten years after his death. Riemann had arrived at the expression

$$_{RL}D^{-\alpha}\phi(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{\phi(\tau)}{(x-\tau)^{1-\alpha}}d\tau, \quad x>0 \tag{14}$$

for fractional integration. Furthermore, we have the most useful forms of left-hand and right-hand Riemann- Liouville (RL) derivatives defined as follows

$$
\begin{aligned}
{}_{RL}D_{a,x}^{\alpha} f(x) &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-\tau)^{m-\alpha-1} f(\tau)\, d\tau, \\
{}_{RL}D_{x,b}^{\alpha} f(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b (\tau-x)^{m-\alpha-1} f(\tau)\, d\tau,
\end{aligned}
\tag{15}
$$

where $m-1 \le \alpha < m$, *a, b* are the terminal points of the interval $[a,b]$, which can also be $-\infty, \infty$. The definition (15) of the fractional differentiation of Riemann-Liouville type leads a conflict between the well-established and polished mathematical theory and proper needs, such as the initial problem of the fractional differential equation, and the nonzero problem related to the Riemann-Liouville derivative of a constant, and so on. A certain solution to this conflict was proposed by Caputo first in his paper [38] (1967) .Caputo's definitions can be written as

$$
\begin{aligned}
{}_{C}D_{a,x}^{\alpha} f(x) &= \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \\
{}_{C}D_{x,b}^{\alpha} f(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b (\tau-x)^{m-\alpha-1} f(\tau)\, d\tau,
\end{aligned}
\tag{16}
$$

where $m-1 \le \alpha < m \in \mathbb{Z}^+$. Obviously, the Caputo derivative is more strict than Riemann-Liouville derivative, one reason is that the *m*-th order derivative is required to exist. The Caputo and Riemann-Liouville formulation coincide when the initial conditions are zero. Besides, the RL derivative is meaningful under weaker smoothness requirements. Also, the RL derivative can be presented as:

$$
{}_{RL}D_{x}^{\alpha} f(x) = D^n D_{a,x}^{\alpha-n} f(x), \quad \alpha \in [n-1, n), \tag{17}
$$

and the Caputo derivative

$$
{}_{C}D_{a,x}^{\alpha} f(x) = D_{a,x}^{\alpha-n} D^n f(x), \quad \alpha \in n(n-1, n), \tag{18}
$$

where $n \in Z^+, D^n$ is the classical n-order derivative. Moreover, previous expressions show that the fractional-order operators are *global* operators having a memory of all past events, making them adequate for modeling hereditary and memory effects in most materials and systems. Also, for the RL derivative, we have

$$
\lim_{\alpha \to (n-1)^+} {}_{a}^{RL}D_t^{\alpha} x(t) = \frac{d^{n-1}x(t)}{dt^{n-1}} \quad \text{and} \quad \lim_{\alpha \to n^-} {}_{a}^{RL}D_t^{\alpha} x(t) = \frac{d^n x(t)}{dt^n} \tag{19}
$$

But for the Caputo derivative, we have

$$
\lim_{\alpha \to (n-1)^+} {}_{a}^{C}D_t^{\alpha} x(t) = \frac{d^{n-1}x(t)}{dt^{n-1}} - D^{(n-1)}x(a) \quad \text{and} \quad \lim_{\alpha \to n^-} {}_{a}^{C}D_t^{\alpha} x(t) = \frac{d^n x(t)}{dt^n} \tag{20}
$$

Obviously, ${}_{RL}D_a^{\alpha}$, $n \in (-\infty, +\infty)$ varies continuously with n, but the Caputo derivative cannot do this. On the other side, initial conditions of fractional differential equations with Caputo derivative have a clear physical meaning and Caputo derivative is extensively used in real applications. On the other side, Grunwald [39] (in 1867) and Letnikov [40] (in 1868) developed an approach to fractional differentiation based on the definition

$$ {}_{GL}D_x^{\alpha} f(x) = \lim_{h\to 0} \frac{\left(\Delta_h^{\alpha} f\left(x\right)\right)}{h^{\alpha}}, \quad \Delta_h^{\alpha} f\left(x\right) = \sum_{0\le|j|<\infty} \left(-1\right)^{|j|} \binom{\alpha}{j} f\left(x - jh\right), \ h>0, \tag{21} $$

which is the left Grunwald-Letnilov (GL) derivative as a limit of a fractional order backward difference. Similarly, we have the right one as

$$ {}_{GL}D_x^{\alpha} f(x) = \lim_{h\to 0} \frac{\left(\Delta_{-h}^{\alpha} f\left(x\right)\right)}{h^{\alpha}}, \quad \Delta_{-h}^{\alpha} f\left(x\right) = \sum_{0\le|j|<\infty} \left(-1\right)^{|j|} \binom{\alpha}{j} f\left(x + jh\right), \ h<0, \tag{22} $$

Therefore, one can define the new form of Grunwald-Letnikov derivative as follows

$$ {}_{GL}D_x^{\alpha} f(x) = \frac{1}{2\cos\left(\dfrac{\pi\alpha}{2}\right)} \lim_{h\to 0} \frac{\left(\Delta_h^{\alpha} f + \Delta_{-h}^{\alpha} f\right)\left(x\right)}{|h|^{\alpha}}, \tag{23} $$

which is called the *Grunwald-Letnikov-Riesz derivative*. As indicated above, the previous definition of GL is valid for α > 0 (fractional derivative) and for α < 0 (fractional integral) and, commonly, these two notions are grouped into one single operator called *differintegral*. The GL derivative and RL derivative are equivalent if the functions they act on are sufficiently smooth. For numerical calculation of fractional–order differ-integral operator one can use relation derived from the GL definition.

$$ {}_{(x-L)}D_x^{\pm\alpha} f(x) \approx h^{\mp\alpha} \sum_{j=0}^{N(x)} b_j^{(\pm\alpha)} f(x - jh) \tag{24} $$

where L is the "memory length", h is the step size of the calculation,

$$ N(t) = min\left\{\left[\frac{x}{h}\right], \left[\frac{L}{h}\right]\right\}, \tag{25} $$

$[x]$ is the integer part of x and $b_j^{(\pm\alpha)}$ is the binomial coefficient given by

$$ b_0^{(\pm\alpha)} = 1, \quad b_j^{(\pm\alpha)} = \left(1 - \frac{1\pm\alpha}{j}\right) b_{j-1}^{(\pm\alpha)} \tag{26} $$

For convenience, Laplace domain is usually used to describe the fractional integro-differential operation for solving engineering problems. The formula for the Laplace transform of the RL fractional derivative has the form:

$$ \int_0^{\infty} e^{-sx} \, {}_{RL}D_{0,x}^{\alpha} f(x) dx = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \, {}_{RL}D_{0,x}^{\alpha-k-1} f(x)\big|_{x=0} \tag{27} $$

Where for $\alpha < 0$ (*i.e.*, for the case of a fractional integral) the sum in the right-hand side must be omitted). Also, Laplace transform of the Caputo fractional derivative is:

$$ \int_0^{\infty} e^{-st} \, {}_0D_t^{\alpha} f(t) dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha < n \tag{28} $$

which implies that all the initial values of the considered equation are presented by a set of only classical integer-order derivatives. Besides that, a geometric and physical interpretation of fractional integration and fractional differentiation can be found in Podlubny's work [41].

3. Preliminaries on integer time-delay systems

A linear, multivariable time-delay system can be represented by differential equation:

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-\tau) \tag{29}$$

and with associated function of initial state:

$$x(t) = \psi_x(t), \qquad -\tau \le t \le 0, \tag{30}$$

Equation (29) is referred to as homogenous state equation. Also, more general a linear, multivariable time-delay system can be represented by following differential equation:

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t) + B_1 u(t-\tau), \tag{31}$$

and with associated function of initial state and control:

$$\begin{aligned} x(t) &= \psi_x(t), \qquad -\tau \le t \le 0, \\ u(t) &= \psi_x(t), \end{aligned} \tag{32}$$

Equation (31) is referred to as nonhomogenous or the unforced state equation, $x(t)$ is state vector, $u(t)$ control vector, A_0, A_1, B_0 and B_1 are constant system matrices of appropriate dimensions, and τ is pure time delay, $\tau = const.$ ($\tau > 0$). Moreover, here it is considered a class of non-linear system with time delay described by the state space equation:

$$\frac{d\mathbf{x}(t)}{dt} = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t-\tau) + B_0 \mathbf{u}(t) + B_1 \mathbf{u}(t-\tau) + \sum_{i=1}^{n} f_i\left(\mathbf{x}(t)\right) + \sum_{j=1}^{m} f_j\left(\mathbf{x}(t-\tau)\right), \tag{33}$$

with the initial functions (32) of the system. Vector functions f_i, f_j, $i = 1, n, j = 1, m$ present nonlinear parameter perturbations of system in respect to $\mathbf{x}(t)$ and $\mathbf{x}(t-\tau)$ respectively. Also, it is introduced next assumption that:

$$\begin{aligned} &\left\| f_i(\mathbf{x}(t)) \right\| \le c_i \left\| \mathbf{x}(t) \right\|, \; i = 1, n \quad t \in \left[0, \infty\right) \\ &\left\| f_j(\mathbf{x}(t-\tau)) \right\| \le c_j \left\| \mathbf{x}(t-\tau) \right\|, \; j = 1, m, \; t \in \left[0, \infty\right) \end{aligned} \tag{34}$$

where $c_i, c_j \in R^+$ are known real positive numbers. Moreover, a linear multivariable time-varying delay system can be represented by differential equation

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t - \tau(t)) + B_0 u(t), \tag{35}$$

and with associated function of initial state

$$x(t) = \psi_x(t), \qquad -\tau_M \le t \le 0. \tag{36}$$

where $\tau(t)$ is an unknown time–varying parameter which satisfies

$$0 \le \tau(t) \le \tau_M, \quad \forall t \in J, \; J = \left[t_o, t_o + T\right], J \subset R \tag{37}$$

Moreover, here it is considered a class of perturbed non-linear system with time delay described by the state space equation

$$\frac{dx(t)}{dt} = \left(A_0 + \Delta A_0\right)x(t) + \left(A_1 + \Delta A_1\right)x(t-\tau(t)) + B_0 u(t) + f_0\left(x(t), x(t-\tau(t))\right), \quad (38)$$

with the given initial functions of the system and vector function f_0.Vector function f_0 present nonlinear parameter perturbations of system in respect to $x(t)$ and $x(t-\tau(t))$ respectively and matrices $\Delta A_0, \Delta A_1$ present perturbations of system, too. Also, it is assumed that next assumption is true.

$$\left\| f_0(x(t), x(t-\tau(t))) \right\| \le c_0 \left\| x(t) \right\| + c_1 \left\| x(t-\tau(t)) \right\|, \; t \in \left[0, \infty\right), \quad (39)$$

where $c_0, c_1 \in R^+$ are known real positive numbers. Dynamical behavior of system (29),(31) or (33) with initial functions (30),or (32) is defined over time interval $J = \left\{t_o, t_o + T\right\}$, where quantity T may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously. It is obvious that $J \in R$. Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded. Let index $"\varepsilon"$ stands for the set of all allowable states of system and index $"\delta"$ for the set of all initial states of the system, such that the set $S_\delta \subseteq S_\varepsilon$. In general, one may write:

$$S_\rho = \left\{ \boldsymbol{x} : \left\| \boldsymbol{x}(t) \right\|_Q^2 < \rho \right\}, \; \rho \in \left[\delta, \varepsilon\right], \quad (40)$$

where Q will be assumed to be symmetric, positive definite, real matrix. S_{α_u} denotes the set of the all allowable control actions. Let $\left|\boldsymbol{x}\right|_{(.)}$ be any vector norm (e.g., $. = 1, 2, \infty$) and $\left\|(.)\right\|$ the matrix norm induced by this vector. Matrix measure has been widely used in the literature when dealing with stability of time delay systems. The matrix measure μ for any matrix $A \in C^{n \times n}$ is defined as follows:

$$\mu\left(A\right) = \lim_{\omega \to 0} \frac{\left\| I + \omega A \right\| - 1}{\omega} \quad (41)$$

The matrix measure defined in (36) can be subdefined in three different ways, depending on the norm utilized in its definitions,[42].

$$\mu_1\left(A\right) = \max_k \left(\operatorname{Re}\left(a_{kk}\right) + \sum_{\substack{i=1 \\ i \ne k}}^{n} \left|a_{ik}\right| \right), \quad (42)$$

$$\mu_2\left(A\right) = \max_k \left(\operatorname{Re}\left(a_{kk}\right) + \sum_{\substack{i=1 \\ i \ne k}}^{n} \left|a_{ik}\right| \right), \quad (43)$$

and $$\mu_\infty(A) = \max_i \left(\operatorname{Re}(a_{ii}) + \sum_{\substack{i=1 \\ i\neq k}}^{n} |a_{ki}| \right) \tag{44}$$

Expression (32) can be written in it's general form as:

$$\begin{aligned} \mathbf{x}(t_o+\theta) &= \psi_x(\theta), \quad -\tau \le \theta \le 0, \ \psi_x(\theta) \in C[-\tau,0] \\ \mathbf{u}(t_o+\theta) &= \psi_u(\theta), \quad -\tau \le \theta \le 0, \ \psi_u(\theta) \in C[-\tau,0] \end{aligned}, \tag{45}$$

where t_o is the initial time of observation of the system (29) and $C[-\tau,0]$ is a Banach space of continuous functions over a time interval of length τ, mapping the interval $[t-\tau,t]$ into R^n with the norm defined in the following manner:

$$\|\psi\|_C = \max_{-\tau\le\theta\le 0} \|\psi(\theta)\|, \tag{46}$$

It is assumed that the usual smoothness condition is present so that is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data.

3.1 Some previous results related to integer time-delay systems

The existing methods developed so far for stability check are mainly for integer-order systems.

Definition 1: *System given by (31) with* $\mathbf{u}(t-\tau) \equiv 0,\ \forall t$ *,satisfying initial condition (4) is finite stable w.r.t* $\{\varsigma(t), \varepsilon, \alpha_u, \tau, J, \mu(A_0) \neq 0\}$, *if and only if:*

$$\psi_x \in S_\delta,\ \forall t \in [-\tau, 0] \tag{47}$$

and

$$\mathbf{u}(t) \in S_{\alpha_u},\ \forall t \in J \tag{48}$$

imply:

$$\mathbf{x}(t;t_0,\mathbf{x_0}) \in S_\varepsilon, \quad \forall t \in [0,T] \tag{49}$$

Illustration of preceding definition is pictured on Fig. 1.

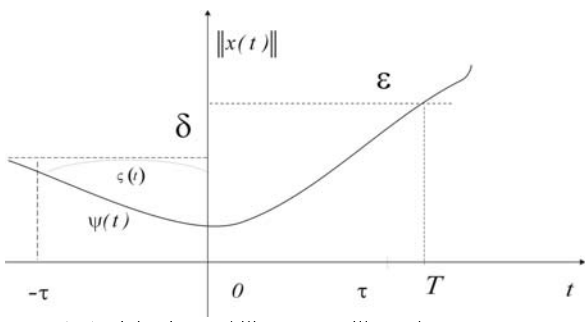

Fig.1 Finite time stability concept illustration

Definition 2: *System given by (31) satisfying initial condition (32) is finite stable w.r.t* $\left\{\delta,\varepsilon,\alpha_{\psi},\alpha_u,\tau,J,\mu\left(A_0\right)\neq 0\right\}$, *if and only if:*

$$\psi_x \in S_{\delta},\ \forall t \in \left[-\tau, 0\right] \tag{50}$$

$$\psi_u \in S_{\alpha_0},\ \forall t \in \left[-\tau, 0\right] \tag{51}$$

and

$$\boldsymbol{u}\left(t\right) \in S_{\alpha_u},\ \forall t \in J \tag{52}$$

imply:

$$\boldsymbol{x}(t;t_0,\boldsymbol{x_0},\boldsymbol{u}(t)) \in S_{\varepsilon},\quad \forall t \in J \tag{53}$$

Theorem 1. *System given by (31), with initial function (32) is finite time stable w.r.t* $\left\{\delta,\varepsilon,\alpha_{\psi},\alpha_u,\tau,J,\mu\left(A_0\right)\neq 0\right\}$, *if the following condition is satisfied*,[43]:

$$\mu_2^{-1}(A_0)e^{\mu_2(A_0)t} < \left(\varepsilon/\delta\right)\sigma^{-1} \tag{54}$$

where:

$$\sigma = a_1\left(\mu_2\left(A_0\right)a_1^{-1} + \left(1-e^{-\mu_2\left(A_0\right)\tau}\right)c_1 + \left(1-e^{-\mu_2\left(A_0\right)t}\right)c_2\right) \tag{55}$$

$$c_2 = \gamma\left(b_0+b_1\right), c_1 = 1 + b_1\left(\gamma+\gamma_{\psi}\right) \tag{56}$$

$$\gamma = \alpha_u/\varepsilon,\ \gamma_{\psi} = \alpha_{\psi}/\varepsilon, a_1 = \left\|A_1\right\|,\ \ b_1 = \left\|B_1\right\|/a_1,\ b_0 = \left\|B_0\right\|/a_1 \tag{57}$$

Results that will be presented in the sequel enables one to check finite time stability of the nonautonomous system to be considered (29),(31) or (33) and (30),(32) without finding the fundamental matrix or corresponding matrix measure.

Definition 3: *System given by (31) satisfying initial condition (32) is finite stable w.r.t* $\left\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\right\}$, $\delta<\varepsilon$ *if and only if:*

$$\left\|\psi_x\right\|_C < \delta,\ \left\|\psi_u\right\|_C < \alpha_0, \tag{58}$$

$$\left\|\boldsymbol{u}(t)\right\| < \alpha_u \quad , \qquad \forall t \in J \tag{59}$$

imply:

$$\left\|\boldsymbol{x}(t)\right\| < \varepsilon \quad , \qquad \forall t \in J \tag{60}$$

Theorem 2. *Nonautonomous system given by (31) satisfying initial condition (33) is finite time stable w.r.t.* $\left\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\right\}$, $\delta<\varepsilon$, *if the following condition is satisfied*,[44]:

$$\left[1+\sigma_{\max}^{A}\left(t-t_0\right)\right]\cdot e^{\sigma_{\max}^{A}(t-t_0)} + \gamma_1^{*}(t-t_0) + \gamma_0^{*}\tau \le \varepsilon/\delta, \qquad \forall t \in J. \tag{61}$$

where

$$\gamma_1^{*} = \gamma_1/\delta,\ \gamma_0^{*} = \gamma_0/\delta,\ \ \gamma_1 = \left(b_0+b_1\right)\alpha_u,\ \gamma_0 = \left(\alpha_0-\alpha_u\right)b_1, \tag{62}$$

4. Preliminaries on stability of fractional order systems including time-delays

In the field of fractional-order control systems, there are many challenging and unsolved problems related to stability theory such as robust stability, bounded input – bounded output stability, internal stability, root-locus, robust controllability, robust observability, etc. In engineering, the fractional order α is often less than 1, so we restrict $\alpha \in (0,1)$ as usual. Even if $\alpha > 1$, we can translate the fractional systems into systems with the same fractional order which lies in $(0,1)$ provided some suitable conditions are satisfied [45]. To demonstrate the advantage of fractional calculus in characterizing system behavior,here, stability properties, let us consider the following illustrative example, [46].

Example 1: Compare the following two systems with initial condition $x(0)$ for $0 < \alpha < 1$,

$$\frac{d}{dt}x(t) = \nu t^{\nu-1}, \qquad {}_C D_{0,t}^{\alpha} x(t) = \nu t^{\nu-1},\ 0 < \alpha < 1. \tag{63}$$

The analytical solutions of previous systems are $t^{\nu} + x(0)$ and $\frac{\nu\Gamma(\nu)t^{\nu+\alpha-1}}{\Gamma(\nu+\alpha)} + x(0)$, respectively. One may conclude, the integer-order system is unstable for any $\nu \in (0,1)$. However, the second,given fractional dynamic system is stable as $0 < \nu < \alpha - 1$, which implies that fractional-order system may have additional attractive feature over the integer-order system. Also, in [47], Tarasov proposed that stability is connected to motion changes at fractional changes of variables where systems which are unstable "*in sense of Lyapuov'*" can be stable with respect to fractional variations. In 1996, Matignon [48] studied the following fractional differential system involving Caputo derivative

$${}_C D_{0,t}^{\alpha} = \frac{d^{\alpha}x}{dt^{\alpha}} = Ax(t), \quad x(0) = x_0,\ \alpha \in (0,1) \tag{64}$$

where $x = (x_1, x_2, ..., x_n)^T$ with initial value $x_0 = (x_{10}, x_{20}, ..., x_{n0})^T$, $A \in R^{n\times n}$ .The stability of the equilibrium of system (64) was first defined and established by Matignon as follows.

Definition 4. *The autonomous fractional order system (64) is said to be*

(a) *stable iff for any* x_0, *there exists* $\varepsilon > 0$ *such that* (65)

$$\|x\| \le \varepsilon \quad \text{for} \quad t \ge 0$$

(b) *asymptotically stable iff* $\lim_{t\to\infty} \|x(t)\| = 0$ (66)

Also,Matignon [48] proposed definition of the BIBO stability for fractional differential system.

Definition 5. *An input/output linear fractional system (67)*

$$\begin{aligned} &\frac{d^{\alpha}x}{dt^{\alpha}} = Ax + Bu,\ x(0) = x_0 \\ &y = Cx \end{aligned} \tag{67}$$

$x \in R^n, y \in R^p$ is *externally stable* or *bounded-input bounded-output (BIBO) iff* $\forall u \in L^{\infty}(R^+, R^m),\quad y = h * u \in L^{\infty}(R^+, R^p)$ *which is equivalent to*: $h \in L^1(R^+, R^{p\times m})$.

Also, in [49] authors give two definitions of the stability for differential systems with the Caputo derivative and Riemann-Liouville derivative, respectively. Besides, the asymptotical stability of higher-dimensional linear fractional differential systems with the Riemann-Liouville fractional order and Caputo fractional order were studied where the asymptotical stability theorems were also derived.

Definition 6. *The zero solution of the following differential system with the α-th order Caputo derivative in which $0 < \alpha < 1$*

$$ {}_C D_{0,t}^{\alpha} X = AX \tag{68}$$

is said to be:

(*i*) *Stable, if* $\forall \varepsilon > 0, \exists \delta > 0$, *when* $\|X_0\| \le \delta$, *the solution* $X(t)$ *to (68) with the initial condition* $X(t) = X_0$ *satisfies* $\|X(t)\| \le \varepsilon$ *for any* $t \ge 0$. (69)

(*ii*) *Asymptotically stable, if the zero solution to (68) is stable, and it is locally attractive, i.e., there exists a* δ_0 *such that* $\|X_0\| \le \delta_0$ *implies that*

$$\lim_{t\to+\infty} \|X(t)\| = 0 \tag{70}$$

Definition 7. *The zero solution of the following differential system with the α-th order Riemann- Liouville derivative in which* $0 < \alpha < 1$

$$ {}_{RL} D_{0,t}^{\alpha} X = AX \tag{71}$$

is said to be:

(*i*) Stable, if $\forall \varepsilon > 0, \exists \delta > 0$, when $\|X_0\| \le \delta$, the solution $X(t)$ to (71) with the initial condition $\left[{}_{RL} D_{0,t}^{\alpha-1} X(t) \right]_{t=0} = X_0$ satisfies

$$\|X(t)\| \le \varepsilon \text{ for any } t \ge 0. \tag{72}$$

(*ii) Asymptotically stable, if the zero solution to (71) is stable, and it is locally attractive, i.e., there exists a* δ_0 *such that* $\|X_0\| \le \delta_0$ *implies that*

$$\lim_{t\to+\infty} \|X(t)\| = 0 \tag{73}$$

Next, one may study the stability of fractional differential systems in two spatial dimensions, and then study the fractional differential systems with higher dimensions. Now, it is studied the fractional differential system with the Caputo derivative,

$$ {}_* D_{0,t}^{\alpha} X = AX, \quad \alpha \in (0,1), \quad A \in R^{n\times n} \tag{74}$$

where fractional derivative ${}_* D_{0,t}^{\alpha}(..) = {}_C D_{0,t}^{\alpha}(..)$ *or* ${}_{RL} D_{0,t}^{\alpha}(..)$. They studied the fractional differential system with the Caputo derivative,as follows:

$$ {}^C D_{0,t}^{\alpha} X = AX, \quad \alpha \in (0,1), \quad A \in R^{n\times n} \tag{75}$$

Theorem 3. *If the real parts of all the eigenvalues of* **A** *are negative, then the zero solution to system (75) is asymptotically stable.*

Also for fractional differential system with the Riemann-Liouville derivative

$$ {}^{RL} D_{0,t}^{\alpha} X = AX, \quad \alpha \in (0,1), \quad A \in R^{n\times n} \tag{76}$$

they stated following theorem.

Theorem 4. *If the real parts of all the eigenvalues of* **A** *are negative, then the zero solution to system (76) is asymptotically stable.*

A fractional-order linear time invariant system can be represented in the following pseudostate space form:

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{77}$$

where the notation d^{α}/dt^{α} indicates the Caputo fractional derivative of fractional commensurate order α, $x \in R^{n}, u \in R^{m}$ and $y \in R^{p}$ are pseudo-state, input, and output vectors of the system, respectively, and $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$. It is worth mentioning that the state space form Eq. (77) is a pseudo-representation because the knowledge of vector x at time $t = t_0$ and input vector $u(t)$ for $t \geq t_0$ are not entirely sufficient to know the behavior of system (77) for $t > t_0$. A fractional-order model is in fact infinite dimensional, therefore its true state vector should be also infinite dimensional.

Theorem 5[48]: *The following autonomous system*,(64)

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t), \; x(t_0) = x_0, \quad 0 < \alpha \leq 1 \tag{78}$$

$x \in R^{n}$ *, and A is an* $n \times n$ *matrix, is asymptotically stable if and only if* $|arg(\lambda)| > \alpha\pi/2$ *is satisfied for all eigenvalues* (λ) *of matrix A. In this case, each component of the states decays toward 0 such as* $t^{-\alpha}$*. Also, this system is stable if and only if* $|arg(\lambda)| > \alpha\pi/2$ *is satisfied for all eigenvalues* (λ) *of matrix A with those critical eigenvalues satisfying* $|arg(\lambda)| = \alpha\pi/2$ *have geometric multiplicity of one.*

Demonstration of this theorem is based on the computation of state vector of system $\|x(t)\| < Nt^{-\alpha}, t > 0, \alpha > 0.$ response to non-zero initial conditions. However, this result remains valid whatever the definition used given that for a linear system without delay, an autonomous system with non-zero initial conditions can be transformed into a non-autonomous system with null initial condition. Also, the stable and unstable regions for $0 < \alpha \leq 1$ is shown in Fig. 2 and they denote the stable and unstable regions for $0 < \alpha \leq 1$ by C_{α}^{-} and C_{α}^{+}, respectively.

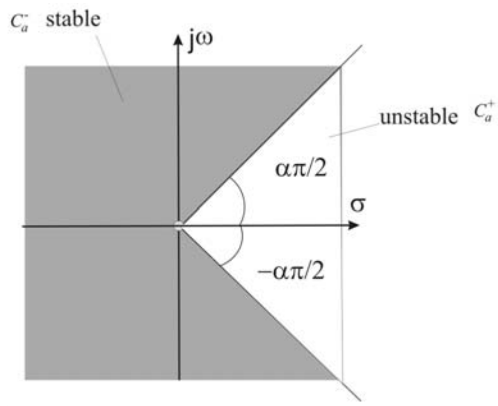

Fig. 2 *Stability region of fractional-order linear time invariant system with order* $0<\alpha\le 1$

For a minimal realization of (77), Matignon has also demonstrated the following theorem,[48].

Theorem 6. *In [48], consider a system given by the following linear pseudostate space form with inner dimension n*:

$$\begin{aligned}&\frac{d^{\alpha}x(t)}{dt^{\alpha}}=Ax(t)+Bu(t),\quad x(0)=x_0\\&y(t)=Cx(t)\end{aligned}\tag{79}$$

where $0<\alpha\le 1$. *Also, assume that the triplet (A,B,C) is minimal. System (79) is bounded-input bounded-output (BIBO) stable if and only if* $\left|\arg(\lambda)>\alpha\pi/2\right|$ *is satisfied for all eigenvalues* λ *of matrix A. When system (79) is externally stable, each component of its impulse response behaves like* $t^{-1-\alpha}$ *at infinity.*

Exponential stability thus cannot be used to characterize asymptotic stability of fractional systems. A new definition is introduced.

Definition 8. $t^{-\gamma}$ *stability*

Trajectory $x(t)$ = 0 of system $d^{\alpha}x(t)/dt^{\alpha}=f(t,x(t))$ (unforced system) is $t^{-\gamma}$ asymptotically stable if the uniform asymptotic stability condition is met and if there is a positive real γ such that:

$$\forall\left\|x(t_0)\right\|\le c,\ \exists Q(x(t_o))\ \text{such that}\ \forall t\ge t_0,\ \left\|x(t)\right\|\le Qt^{-\gamma}\tag{80}$$

$t^{-\gamma}$ stability will thus be used to refer to the asymptotic stability of fractional systems.As the components of the state $x(t)$ slowly decay towards 0 following $t^{-\gamma}$, fractional systems are sometimes called *long memory systems*.

5. Stability of fractional delay system

In spite of intensive researches, the stability of fractional order (time delay) systems remains an open problem. As for linear time invariant integer order systems, it is now well known that stability of a linear fractional order system depends on the location of the system poles in the complex plane. However, poles location analysis remains a difficult task in the general case. For commensurate fractional order systems, powerful criteria have been proposed. The most well known is Matignon's stability theorem [48]. It permits us to check the system stability through the location in the complex plane of the dynamic matrix eigenvalues of the state space like system representation. Matignon's theorem is in fact the starting point of several results in the field. As we know, due to the presence of the exponential function $e^{-\tau s}$, this equation has an infinite number of roots, which makes the analytical stability analysis of a time-delay system extremely difficult. In the literature few theorems are available for stability testing of fractional-delay systems. Almost all of these theorems are based on the locations of the transfer function poles [24,50] and since there is no universally applicable analytical method for solving fractional-delay equations in s domain, the numerical approach is commonly used. In the field of infinite-dimensional fractional-delay systems most studies are concerned about the stability of a class of distributed systems whose transfer functions involve $\sqrt{s}$ and/or $e^{-\sqrt{s}}$,[51]. Many examples of fractional differential systems with delay can be found in the literature. Simple examples such as $G(s)=\exp(-a\sqrt{s})/s,\ a>0$ arising in theory of transmission lines [52], or one can find in [53] fractional delay systems with transfer function of linked to the heat equation which leads to transfer functions $G(s)$ such as

$$G(s)=\frac{\cosh(x\sqrt{s})}{\sqrt{s}\,\sinh(\sqrt{s})},\ \left(0\le x\le 1\right)\ \text{ or }\ G(s)=\frac{2e^{-a\sqrt{s}}}{b(1-e^{-2a\sqrt{s}})} \tag{82}$$

For example, Hotzel [54] presented the stability conditions for fractional-delay systems with the characteristic equation $\left(as^{\alpha}+b\right)+\left(cs^{\alpha}+d\right)e^{-\rho s}=0$. Chen and Moore [22] analyzed the stability of a class of fractional-delay systems whose characteristic function can be represented as the product of factors of the form $\left(as+b\right)^{r}e^{cs}+d=0$ where the parameters a,b,c,d ,and r are all real numbers. In fact, they computed the characteristic roots of the system using the Lambert W function, which has become a standard library function of many mathematical software. In other words, they got a stability condition of (83), given by a transcendent inequality via the Lambert function [22,55]. They considered the following delayed fractional equation

$$\frac{d^{q}y(t)}{dt^{q}}=K_{p}y(t-\tau) \tag{83}$$

where q and Kp are real numbers and $0 < q < 1$, time delay τ is a positive constant and all the initial values are zeros. We are interested in telling whether the system (10) is stable or not for a given set of combination of the three parameters: q, Kp and τ. The stability condition is that for all possible q , r and Kp

$$\frac{q}{\tau}W\left(\frac{\tau}{r}\left(K_{p}\right)^{1/q}\right)\le 0 \tag{84}$$

In inequality, $W(.)$ denotes the Lambert function such that $W(x)e^{W(x)}=x$. However, such a bound remains analytic and is difficult to use in practice. In paper [55], the application of Lambert W function to the stability analysis of time-delay systems is re-examined through actually constructing the root distributions of the derived a transcendental characteristic equation's (TCE) of some chosen orders. It is found that the rightmost root of the original TCE is not necessarily a principal branch Lambert W function solution, and that a derived

TCE obtained by taking the *n*th power of the original TCE introduces superfluous roots to the system. Further, Matignon's theorem has been used in [56] to investigate fractional differential systems with multiple delays stability. The proposed stability conditions are based on the root locus of the system characteristic matrix determinant but the proposed conditions are thus difficult to use in practice. Authors used fractional derivative Caputo definition of derivative where by using the Laplace transform, it is introduced a characteristic equation for the above system with multiple time delays. They discovered that if all roots of the characteristic equation have negative parts, then the equilibrium of the above linear system with fractional order is Lyapunov globally asymptotical stable if the equilibrium exist that is almost the same as that of classical differential equations. Namely, the following *n*-dimensional linear fractional differential system with multiple time delays:

$$\begin{aligned} \frac{d^{q_1}x_1(t)}{d^{q_1}t} &= a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \ldots + a_{1n}x_n(t-\tau_{1n}), \\ \frac{d^{q_2}x_2(t)}{d^{q_2}t} &= a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \ldots + a_{2n}x_n(t-\tau_{2n}), \\ &\ldots\ldots\ldots\ldots\ldots \\ \frac{d^{q_n}x_n(t)}{d^{q_n}t} &= a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \ldots + a_{nn}x_n(t-\tau_{nn}), \end{aligned} \tag{85}$$

where q_i is real and lies in (0,1), the initial values $x_i(t) = \phi_i(t)$ are given for$-\max_{i,j} \tau_{ij} = -\tau_{\max} \le t \le 0$ and $i = 1,2,...,n$. In this system, time-delay matrix $T = (\tau_{ij})_{n\times n} \in (R^+)^{n\times n}$, coefficient matrix $A = (a_{ij})_{n\times n}$, state variables $x_i(t), x_i(t-\tau_{ij}) \in R$, and initial values $\phi_i(t) \in C^0[-\tau_{max}, 0]$. Its fractional order is defined as $q = (q_1, q_2, ..., q_n)$. If $q_i = q_j$ and $\tau_{ij} = 0,\ i,j = 1,2,...,n$, then system (85) is actually the one considered in [56].

$$\Delta(s) = \begin{pmatrix} s^{q_1} - a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \ldots & -a_{1n}e^{-s\tau_{1n}} \\ -a_{21}e^{-s\tau_{21}} & s^{q_2} - a_{22}e^{-s\tau_{22}} & \ldots & -a_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \ldots & s^{q_n} - a_{nn}e^{-s\tau_{nn}} \end{pmatrix} \tag{86}$$

where $\Delta(s)$ denotes a *characteristic matrix* of system (1) and $det(\Delta(s))$ a *characteristic polynomial* of (86). The distribution of $det(\Delta(s))$'s eigenvalues totally determines the stability of system (86).

Theorem 7. *If all the roots of the characteristic equation* $det(\Delta(s)) = 0$ *have negative real parts, then the zero solution of system (1) is Lyapunov globally asymptotically stable. If* $n = 1$*, then (86) is reduced to the system studied in* [56].

Bonnet and Partington [23,24] analyzes the BIBO stability of fractional exponential delay systems which are of retarded or neutral type. Conditions ensuring stability are given and these conditions can be expressed in terms of the location of the poles of the system. In view of constructing robust BIBO stabilizing controllers, explicit expressions of coprime and B´ezout factors of these systems are determined.Also,they have handled the robust

stabilization of fractional exponential delay systems of retarded type. The determination of coprime and B´ezout factors in the case of neutral systems is under study in both cases.
However, all these contributions do not provide universally acceptable practical effective algebraic criteria or algorithms for testing the stability of a given general fractional delay system. Although the stability of the given general characteristic equation can be checked with the Nyquist criterion or the Mikhailov criterion, it becomes sufficiently difficult when a computer is used since one should find an angle of turn of the frequency response plot for an infinite variation of the frequency ω . A visual conclusion on stability with respect to the constructed part of the plot is not practically reliable, since, along with an infinite spiral, the delay generates loops whose number is infinite. As is evidenced from the literature the lack of universally acceptable algebraic algorithms for testing the stability of the characteristic equation has hindered the advance of control system design for fractional delay systems. This is particularly true in the case of designing fixed-structure fractional-order controller, e.g., $PI^{\alpha}D^{\beta}$. On the other side, Hwang and Cheng [57] proposed a numerical algorithm which use methods that are based on the Cauchy integral theorem and suggested the modified complex integral in the form of

$$J_k = \int_{-i\infty}^{i\infty} \frac{f(s)}{\left(s+h_1+ih_2\right)^k f(ih_2)} ds \tag{87}$$

where $h_1 > 0$ and h_2 are randomly chosen real constants lying in a specified interval and k is a positive integer. The randomness of the parameters h1 and h2 makes the probability of the zero sum of the residues of all poles of the integrand being practically zero. Hence, the stability of a given fractional-delay system can be achieved by evaluating the integral J_k and comparing its value with zero. Also, the proposed algorithm provides no idea about the number and the location of unstable poles. In paper [58], an effective numerical algorithm for determining the location of poles and zeros on the first Riemann sheet is presented. The proposed method is based on the *Rouche's theorem* and can be applied to all multi-valued transfer functions defined on a Riemann surface with finite number of Riemann sheets where the origin is a branch point. This covers all practical (finite-dimensional) fractional-order transfer functions and *fractional-delay systems*.

5.1 Finite time stability and stabilization of fractional order time delay systems

As we know, the boundedness properties of system responses are very important from the engineering point of view. That is to say, enable system trajectories to stay within a priori given sets for the fractional order time-delay systems in state-space form, i.e., system stability from the non-Lyapunov point of view is considered. From this fact and our the best knowledge, we firstly introduced and defined finite-time stability for fractional order time delay systems [26-27, 60,62-63]. We also need the following definitions to analyze the case of fractional order systems with time-delay from non-Lyapunov point of view. First, we introduce the same order fractional differential system with time-delay (88) as well as multiple time delays (90) represented by the following differential equations:

$$_{*}D_{to,t}^{\alpha}x(t) = \frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}} = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B_0u(t), \quad 0<\alpha<1, \tag{88}$$

with the associated function of initial state:

$$\mathbf{x}(t_0+t)=\psi_x(t)\in C\left[-\tau,0\right],\quad -\tau\le t\le 0. \tag{89}$$

Moreover, it is shown in [26] that fractional-order time delay state space model of PD^{α} control of Newcastle robot can be presented by (88) in state space form. Here, ${}_{*}D^{\alpha}_{to,t}(.)$ denotes either Caputo fractional derivative ${}_{C}D^{\alpha}_{to,t}(.)$ or Riemann-Liouville fractional derivative ${}_{RL}D^{\alpha}_{to,t}(.)$. Also, fractional differential system with multiple time delays can be presented as follows:

$$\begin{aligned}&{}_{*}D^{\alpha}_{to,t}x(t)=\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}}=A_0\mathbf{x}(t)+\sum_{i=1}^{n}A_i\mathbf{x}(t-\tau_i)+B_0u(t),\quad 0<\alpha<1,\\&0\le\tau_1<\tau_2<...<\tau_i<...<\tau_m=\Delta\end{aligned} \tag{90}$$

with the associated function of initial state:

$$x(t_0+t)=\psi_x(t)\in C\left[-\Delta,0\right],\quad -\tau\le t\le 0. \tag{91}$$

and where $A_i(i=0,1,...,m)$, B_0 are constant system matrices of appropriate dimensions, and $\tau_i>0\ \ (i=1,2,...,m)$ are pure time delays.

Definition 9.[59] *System given by (88), ($u(t)\equiv 0$) satisfying initial condition (89) is finite stable w.r.t* $\left\{t_o,J,\delta,\varepsilon,\tau\right\},\ \ \delta<\varepsilon$ *if and only if:*

$$\left\|\psi_x\right\|_C<\delta, \tag{92}$$

implies:

$$\left\|\mathbf{x}(t)\right\|<\varepsilon\quad,\qquad\forall t\in J, \tag{93}$$

Definition 10.[59] *System given by (90), ($u(t)\equiv 0$) satisfying initial condition (91) is finite stable w.r.t* $\left\{t_o,J,\delta,\varepsilon,\Delta\right\},\ \ \delta<\varepsilon$ *if and only if*:

$$\left\|\psi_x\right\|_C<\delta,\ \forall t\in J_\Delta,\quad J_\Delta=\left[-\Delta,0\right]\in R, \tag{94}$$

implies:

$$\left\|\mathbf{x}(t)\right\|<\varepsilon\quad,\qquad\forall t\in J, \tag{95}$$

Definition 11.[27,62]*System given by (90) satisfying initial condition (91) is finite stable w.r.t* $\left\{\delta,\varepsilon,\alpha_u,\Delta,t_o,J,\right\},\ \ \delta<\varepsilon$ *if and only if:*

$$\left\|\psi_x\right\|_C<\delta,\ \forall t\in J_\Delta,\quad J_\Delta=\left[-\Delta,0\right]\in R \tag{96}$$

and

$$\left\|\mathbf{u}(t)\right\|<\alpha_u\quad,\qquad\forall t\in J,\quad \alpha_u>0 \tag{97}$$

imply:

$$\left\|\mathbf{x}(t)\right\|<\varepsilon\quad,\qquad\forall t\in J \tag{98}$$

Also, nonlinear fractional differential system with time delay in state and control can be presented as follows:

$$\begin{aligned}{}_{*}D^{\alpha}_{to,t}x(t)=\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}}&=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)+B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau)+\\&+\sum_{i=1}^{n}f_i\left(\mathbf{x}(t)\right)+\sum_{j=1}^{m}f_j\left(\mathbf{x}(t-\tau)\right),\ 0<\alpha<1,\end{aligned} \tag{99}$$

and with associated function of initial state and control:

$$\boldsymbol{x}(t)=\psi_x(t), \quad \boldsymbol{u}(t)=\psi_u(t), \qquad -\tau \le t \le 0 \tag{100}$$

Equation (99) is referred to as nonlinear nonhomogenous state equation, A_0, A_1, B_0 and B_1 are constant system matrices of appropriate dimensions, and vector functions f_i, f_j, $i=1,n, j=1,m$ present nonlinear parameter perturbations of system in respect to $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t-\tau)$ respectively.

Definition 12: *System given by (99) satisfying initial condition (100) is finite stable w.r.t* $\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\}$, $\delta<\varepsilon$ *if and only if*:

$$\|\psi_x\|_C < \delta, \qquad \|\psi_u\|_C < \alpha_0, \tag{101}$$

$$\|\boldsymbol{u}(t)\| < \alpha_u \quad , \qquad \forall t \in J \tag{102}$$

imply:

$$\|\boldsymbol{x}(t)\| < \varepsilon \quad , \qquad \forall t \in J \tag{103}$$

In what follows, we introduce the sufficient conditions on finite-time stability. In [59], we considered the fractional time-delay systems (88),(90) in the case of $u(t)\equiv 0$.

Theorem 8.(A) *Autonomous system given by (88) satisfying initial condition (89) is finite time stable w.r.t.* $\{\delta,\varepsilon,\tau,t_o,J,\}$, $\delta<\varepsilon$ *, if the following condition is satisfied*:

$$\left[1+\frac{\sigma_{\max}^A\left(t-t_0\right)^\alpha}{\Gamma(\alpha+1)}\right]\cdot e^{\frac{\sigma_{\max}^A (t-t_0)^\alpha}{\Gamma(\alpha+1)}} \le \varepsilon/\delta, \qquad \forall t \in J. \tag{104}$$

where $\sigma_{max}(.)$ *being the largest singular value of matrix (.), namely:*

$$\sigma_{\max}^A=\sigma_{\max}\left(A_0\right)+\sigma_{\max}\left(A_1\right), \tag{105}$$

and $\Gamma(.)$ *is the Euler's gamma function.*

B) *Autonomous system given by (90) satisfying initial condition (91) is finite time stable w.r.t.* $\{\delta,\varepsilon,\Delta,t_o,J,\}$, $\delta<\varepsilon$ *, if the following condition is satisfied*:

$$\left[1+\frac{\sigma_{\Sigma\max}^A\left(t-t_0\right)^\alpha}{\Gamma(\alpha+1)}\right]\cdot e^{\frac{\sigma_{\Sigma\max}^A (t-t_0)^\alpha}{\Gamma(\alpha+1)}} \le \varepsilon/\delta, \qquad \forall t \in J. \tag{106}$$

where $\sigma_{\Sigma\max}^{A}(.)=\sum_{i}\sigma_i\left(A_i\right)$ *of matrices* A_i, $i=0,1,2,...,n$. *where* $\sigma_{max}(.)$ *being the largest singular value of matrix* A_i, $i=0,1,2,...,n$.

The above stability results for linear time-delay fractional differential systems were derived by applying Bellman -Gronwall's inequality. In that way, one can check system stability over finite time interval.

Remark 1[60]**:** If $\alpha=1$, case A, one can obtain same conditions which related to integer order time delay systems (1) as follows:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_{0}\right)^{1}}{1}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_{0})^{1}}{1}}\leq\varepsilon/\delta,\quad\forall t\in J,\ \Gamma(2)=1\tag{107}$$

For the nonautonomous case,Zhang [61] also considered the following initial value problem

$${}_{RL}D_{0,t}^{\alpha}x\left(t\right)=A_{0}x(t)+A_{1}x(t-\tau)+f(t),\quad t\geq0;\quad x\left(t\right)=\phi\left(t\right),\ t\in\left[-\tau,0\right]\tag{108}$$

where $0 < \alpha < 1$, ϕ is a given continuous function on $[-\tau, 0]$, A_0 and A_1 are constant system matrices of appropriate dimensions, and τ is a constant with $\tau > 0$. The system is defined over time interval $J = [0, T]$, where T is a positive number, *f(t)* is a given continuous function on [0, T]. Similarly, the sufficient conditions of finite-time stability were derived by applying Bellman-Gronwall's inequality.

Theorem 9. *System given by (108) satisfying initial condition is finite-time stable w.r.t {0, J, δ, ε, τ}, δ <ε, if the following condition is satisfied:*

$$\frac{\left(M+\mu_{1}\right)\left(t^{\alpha}\right)}{\Gamma\left(\alpha+1\right)}\cdot e^{\frac{\mu_{\max}^{A}(t^{\alpha})}{\Gamma(\alpha+1)}}\leq\varepsilon/\delta,\quad\forall t\in J,\tag{109}$$

Where $M \geq \|f\|/\|\phi\|$, and $\Gamma(.)$ is the Euler's gamma function, $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}\|\phi(\theta)\|$

$\mu_{\max}^{A} = \mu_{\max}(A_0) + \mu_{\max}(A_1)$, $\mu_1 = \mu_{\max}(A_1)$.

In paper [62], we considered a class of fractional non-linear perturbed autonomous system with time delay described by the state space equation:

$$\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}}=\left(A_{0}+\Delta A_{0}\right)\mathbf{x}(t)+\left(A_{1}+\Delta A_{1}\right)\mathbf{x}(t-\tau)+f_{0}\left(\mathbf{x}(t)\right),\tag{110}$$

with the initial functions (89)of the system and vector functions f_0 satisfied (34).

Theorem 10. *Nonlinear perturbed autonomous system given by (110) satisfying initial condition (89) and (34) is finite time stable w.r.t.* $\{\delta,\varepsilon,t_o,J,\}$, $\delta<\varepsilon$, *if the following condition is satisfied:*

$$\left(1+\frac{\mu_{p}(t-t_{0})^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)e^{\frac{\mu_{p}\left(t-t_{0}\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}}\leq\varepsilon/\delta,\ \forall t\in J\text{,}\tag{111}$$

where $\Gamma(.)$ *Euler's gamma function, and* $\mu_{Aoco} = \sigma_{Ao} + \gamma_{\Delta A_0} + c_0$, $\sigma_{A1\Delta} = \sigma_{A1} + \gamma_{\Delta A_1}$,

$\mu_p = \mu_{Aoco} + \sigma_{A1\Delta}, \sigma_{\Delta Ao} \leq \gamma_{\Delta Ao}$, $\sigma_{\Delta A1} \leq \gamma_{\Delta A1}$.

Remark 2: If we have no perturbed system $\Delta\mathbf{A}_0 = 0, \Delta\mathbf{A}_1 = 0, \mathbf{f}_0(\mathbf{x}(\mathrm{t})) = 0$ one can obtain same conditions which related to Theorem 7.

Further, paper [63] presents natural extension of the our paper [59] where it is obtained new stability criteria for nonautonomous fractional order time delay system (88).

Theorem 11. Nonautonomous system given by (88) satisfying initial condition (89) is finite time stable w.r.t. $\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\}$, $\delta<\varepsilon$, if the following condition is satisfied:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}+\gamma^{\bullet}\frac{(t-t_0)^{\alpha}}{\Gamma\left(\alpha+1\right)}\le\varepsilon/\delta,\quad\forall t\in J. \tag{112}$$

where $\gamma^{\bullet}=b_0\alpha_u/\delta,\ \left\|B_0\right\|=b_0$.

Remark 3. If $\alpha=1$, one can obtain same conditions which related to integer order time delay systems (31) , $B_1=0$ as follows, [18]:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^{1}}{1}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^{1}}{1}}+\gamma^{\bullet}\frac{(t-t_0)^{1}}{1}\le\varepsilon/\delta,\quad\forall t\in J,\ \Gamma(2)=1 \tag{113}$$

Moreover, in same paper [63], it is proposed finite time stability criteria for a class of fractional non-linear nonautonomous system with time delay in state and in control as follows:

$$\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}}=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)+B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau)+f_0\left(\mathbf{x}(t)\right)+f_1\left(\mathbf{x}(t-\tau)\right), \tag{114}$$

with the initial functions (99) of the system and vector functions f_0, f_1 satisfied (34).

Theorem 12: Nonlinear nonautonomous system given by (114) satisfying initial condition (99) is finite time stable w.r.t. $\left\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\right\}$, $\delta<\varepsilon$, if the following condition is satisfied: (115)

$$\left(1+\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)e^{\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}+\frac{\gamma^{\bullet}{}_{u0}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\gamma^{\bullet}{}_{u1}\left(t-t_0-\tau\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\gamma^{\bullet}{}_{01}\left(\tau\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\le\varepsilon/\delta,\ \forall t\in J$$

where $\gamma^{\bullet}{}_{u0}=\alpha_u b_0/\delta,\ \gamma^{\bullet}{}_{u1}=\alpha_u b_1/\delta,\ \gamma^{\bullet}{}_{01}=\alpha_0 b_1/\delta$.

Recently, we studied and reported in paper,[27] finite-time stability analysis of linear fractional order single time delay systems where a Bellman-Gronwall`s approach is proposed, using as the starting point a recently obtained *generalized* Gronwall inequality reported in [28].

Theorem 13. The linear nonautonomous system given by (88) satisfying initial condition $x(t)=\psi_x(t),\ -\tau\le t\le 0$ is finite time stable w.r.t. $\left\{\delta,\varepsilon,\alpha_u,J_0,\right\}$, $\delta<\varepsilon$, if the following condition is satisfied:

$$\left(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)E_{\alpha}\left(\sigma_{\max 01}t^{\alpha}\right)+\frac{\gamma^{\bullet}{}_{u0}t^{\alpha}}{\Gamma\left(\alpha+1\right)}\le\varepsilon/\delta,\ \forall t\in J_0=\left\{0,T\right\}, \tag{116}$$

where $\gamma^{\bullet}{}_{u0}=\alpha_u b_0/\delta$, and $\sigma_{max}(.)$ being the largest singular value of matrix (.), where: $\sigma_{\max 01}=\sigma_{\max}\left(A_0\right)+\sigma_{\max}\left(A_1\right)$ and $E_{\alpha}\left(.\right)$ denotes Mittag-Leffler function (see Appendix).

Remark 4. If $\alpha=1$, one can obtain same conditions which related to integer order time delay systems (31), $B_1=0$ as follows [18]:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^1}{1}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^1}{1}}+\gamma\cdot\frac{(t-t_0)^1}{1}\leq\varepsilon/\delta,\quad \forall t\in J,\ \Gamma(2)=1, \tag{117}$$

$$E_{\alpha=1}(z)=e^{z}$$

Theorem 14. The linear autonomous system given by Eq. (88) $B_0=0$, satisfying initial condition $x(t)=\psi_x(t),\ -\tau\leq t\leq 0$ is finite time stable w.r.t. $\{\delta,\varepsilon,J_0,\}$, $\delta<\varepsilon$, if the following condition is satisfied:

$$\left(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha}\left(\sigma_{\max 01}t^{\alpha}\right)\leq\varepsilon/\delta,\ \forall t\in J_0, \tag{118}$$

Remark 5. In same manner, one may conclude that if $\alpha=1$, see (21), it follows same conditions [60],Eq. (107) which relate to integer order time delay systems (29).

5.2 An illustrative example

Using a Time-Delay PD^{α} compensator on a linear system of equations with respect to the small perturbation $e(t)=y(t)-y_d(t)$, one may obtain:

$$\dot{e}(t)+\omega e(t)=K_P e(t-\tau)+K_D de^{(\alpha)}(t-\tau)/dt^{\alpha}+u(t), \tag{119}$$

where: $\alpha=1/2,\omega=2,\ K_p=3,\ K_D=4,u(t)$-feedforward control. Also, all initial values are zeros. Introducing: $x_1(t)=e(t),\ \ x_2(t)=d^{1/2}e(t)/dt^{1/2}$,one may write (119) in state-space form, $\mathbf{x}(t)=\left(x_1,x_2\right)^T$:

$$D_t^{1/2}\mathbf{x}(t)=\begin{bmatrix}0&1\\-2&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}0&0\\3&4\end{bmatrix}\begin{bmatrix}x_1(t-\tau)\\x_2(t-\tau)\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u(t), \tag{120}$$

with an associated function of the initial state: $\mathbf{x}(t)=\psi_x(t)=0,\ \ -\tau\leq t\leq 0.$ Now, we can check the finite time stability *wrt* $\{t_0=0,J=\{0,2\},\delta=0.1,\varepsilon=100,\tau=0.1,\alpha_u=1\}$, where $\psi_x(t)=0,\ \forall t\in[-0.1,0]$. From the initial data and the Eq.(120) it yields:

$$\left\|\psi_x(t)\right\|_C<0.1,\quad \sigma_{\max}\left(A_0\right)=2,\sigma_{\max}\left(A_1\right)=5,\Rightarrow\sigma_{\max 0,1}=7 \tag{121}$$

Applying the condition of the Theorem 13 (116) one can get:

$$\left[1+\frac{7T_e^{0.5}}{0.886}\right]\cdot E_{0.5}\left(7T_e^{0.5}\right)+\frac{10\cdot T_e^{0.5}}{0.886}\leq 100/0.1\quad\Rightarrow T_e\approx 0.1s. \tag{122}$$

T_e being "estimated time" of finite time stability.

Conclusion

In this paper, we have studied and presented the finite time stability of perturbed (non)linear fractional order time delay systems. We have employed the "classical" and the generalization of Gronwall Belmann lemma to obtain finite time stability criteria for proposed class of time delay system. Also, they are presented some basic results on the stability of fractional order time delay systems as well as free delay systems. Finally, a numerical example is given to illustrate the validity of the proposed procedure.

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Appendix

Mittag-Leffler Function

Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solutions of fractional-order systems is the Mittag-Leffler function defined as

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{A1}$$

where $\alpha>0$ and $z\in C$. The Mittag-Leffler function with two parameters appears most frequently and has the following form

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{A2}$$

where $\alpha>0,\ \beta>0$, and $z\in C$. For $\beta=1$ we obtain $E_{\alpha,1}(z)=E_\alpha(z)$ and $E_{1,1}(z)=e^z$

Lemma (*Gronwall Inequality*).

Suppose that $g(t)$ and $\varphi(t)$ are continuous in $[t_0,t_1], g(t)\geq 0, \lambda\geq 0$ and $r\geq 0$ are two constants. If

$$\varphi(t)\leq\lambda+\int_0^t\left[g(s)\varphi(s)+r\right]ds \tag{A3}$$

then
$$\varphi(t)\leq\left(\lambda+r(t_1-t_0)\right)\exp\left(\int_0^t\left[g(s)\right]ds\right),\quad t_0\leq t\leq t_1 \qquad \text{A4)}$$

Theorem A ([28] *Generalized Gronwall inequality*) Suppose $x(t), a(t)$ are nonnegative and local integrable on $0\leq t<T, some\ T\leq+\infty$,and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0\leq t<T$, $g(t)\leq M=const$, $\alpha>0$ with

$$x(t)\leq a(t)+g(t)\int_0^t(t-s)^{\alpha-1}x(s)ds \tag{A5}$$

on this interval.Then

$$x(t)\leq a(t)+\int_0^t\left[\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\right)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]ds,\quad 0\leq t<T \tag{A6}$$

Corollary 2.1 of (Theorem A) [28] Under the hypothesis of Theorem 2.2, let $a(t)$ be a nondecreasing function on $[0,T)$. Then holds:

$$x(t) \le a(t)E_{\alpha}\left(g(t)\Gamma(\alpha)t^{\alpha}\right) \quad \text{(A7)}$$

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