REAL TIME CONTROL OF ROTARY INVERTED PENDULUM

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Summary: *This paper proposes advanced control strategy for a rotary inverted pendulum (RIP). RIP is an underactuated mechanical system because it has only one control input and two degrees of freedom. Because of its complex nonlinear dynamics, RIP is usually used to test performance of different control algorithms. First, laboratory electromechanical system representing the full control system is described in short, followed by the mathematical model for the RIP. Control problem is divided and implemented in two different steps: swing-up and stabilization routines. Here, a partial feedback linearization procedure and PID control are suggested for the control of RIP. The effectiveness of the proposed control method is tested in Matlab Simulink environment.*
Keywords: *inverted pendulum, PID control, feedback linearization*

1. INTRODUCTION

Underactuated systems have more degrees of freedom than actuators, [1-3]. A rotational inverted pendulum, also known as Furuta pendulum, is an example of such a system, [4-7]. Almost all dynamic systems are nonlinear by its nature, therefore a lot of research is done in the area of nonlinear control. The aim of this paper is to develop a nonlinear control system for both the rotational pendulum and actuated arm. First, a description of the Furuta pendulum will be given. Then, a mathematical model of the system will be derived. The control strategy consists of two parts, a swing up and a balancing phase. The theory of inverse dynamic control will be used for the latter. However, the resulting zero dynamics of the actuated arm shows unstable behavior. Hence, a control feedback law will be extended in order to stabilize the horizontal arm.

2. DYNAMIC EQUATIONS OF ROTARY INVERTED PENDULUM

2.1 Description of the system

In Fig. 1 a schematic of Furuta pendulum and real laboratory model are shown.

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Inverted pendulum is a mechanical system with two degrees of freedom, where angular position of the arm and the pendulum are denoted as θ and φ , respectively. The arm is driven with a torque, while no torque is applied directly to the pendulum.

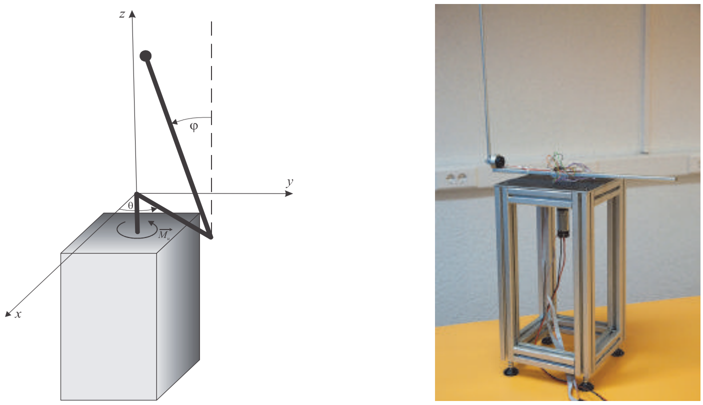

Fig. 1 A schematic of the rotational inverted pendulum and available laboratory model

The variables used to define the model of the rotary inverted pendulum are shown in table below.

Table 1 *Description of the parameters used in the modeling of the system*

m_1 , m_2	mass of the arm and pendulum, respectively
R_1	distance of the arm's pivot point to the pendulum's pivot point
R_2	distance of the pendulum's pivot point to its end (extreme)
$2r_1$, $2r_2$	total length of the arm, and pendulum respectively
$J_{\varsigma 1}$	moment of inertia of the arm with respect to its center of mass
$J_{\xi 2}, J_{\eta 2}, J_{\varsigma 2}$	$J_{\xi 2}, J_{\eta 2}, J_{\varsigma 2}$ – axial moments of inertia of the pendulum with respect to its center of mass

The laboratory electromechanical system is comprised of three subsystems: measurement, power and control supervision. The first one consists of high resolution encoders for measuring arm and pendulum angle positions. The second one provides the power for DC motor which drives the arm shaft. Embedded compact RIO controller, together with monitoring PC forms the last and most important part. Lab VIEW software is used for implementing the real time software and supervision.

2.2 Mathematical model of rotary inverted pendulum

Here, the Rodriguez method is proposed for modeling the dynamics of the system where configuration of the mechanical model can be defined by generalized coordinates q_1 and q_2 , representing θ and φ , respectively. The equations of motion of the inverted pendulum can be expressed in a covariant form of Lagrange's equation of second kind as follows [1,2]:

$$\sum_{\alpha=1}^{n} a_{\gamma\alpha}\ddot{q}_{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}_{\alpha}\dot{q}_{\beta} = Q_{\gamma} \quad \gamma = 1,2 \tag{1}$$

where the coefficients $a_{\alpha\beta}$ are the covariant coordinates of the basic metric tensor $\left[a_{\gamma\alpha}\right] \in R^{2\times 2}$, $\Gamma_{\alpha\beta,\gamma}$ $\alpha,\beta,\gamma = 1,2$ presents Christoffel symbols of the first kind, and Q_{γ} denote the generalized gravitational and control forces.

The equations of motion of our system can be rewritten in compact matrix form:

$$A(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Q}^{a} \tag{2}$$

where $\boldsymbol{q} = (\theta \;\; \varphi)^{T}$, $A(\boldsymbol{q}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is basic metric tensor, $C(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^{2}$ – vector which takes care of configuration of inverted pendulum system and velocity dependent effects, $\boldsymbol{Q}^{g}\left(q\right) = -\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^{2}$ – vector of generalized gravitational forces, and $\boldsymbol{Q}^{a} = \left(M \quad 0\right)^{T} \in \mathbb{R}^{2}$ – vector of generalized control forces. Finally, Eq. (2) written in full form becomes:

$$a_{11}\ddot{\theta} + a_{12}\ddot{\varphi} + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{22,1}\dot{\varphi}^{2} = M \tag{3}$$

$$a_{12}\ddot{\theta} + a_{22}\ddot{\varphi} - \Gamma_{12,1}\dot{\theta}^{2} = Q_{2}^{g} \tag{4}$$

where are

$$\begin{aligned}
&a_{11} = J_{\varsigma 1} + J_{\eta 2}\sin^{2}(\varphi) + J_{\varsigma 2}\cos^{2}(\varphi) + m_{2}R_{1}^{2} + m_{1}(R_{1}-r_{1})^{2} + m_{2}(R_{2}-r_{2})^{2}\sin^{2}(\varphi) \\
&a_{12} = -m_{2}R_{1}(R_{2}-r_{2})\cos(\varphi) = -K_{3}\cos(\varphi), \quad a_{22} = J_{\xi 2} + m_{2}(R_{2}-r_{2})^{2} = K_{4} \\
&\Gamma_{12,1} = 0.5\left(m_{2}(R_{2}-r_{2})^{2} + J_{\eta 2} - J_{\varsigma 2}\right)\sin(2\varphi) = K_{2}\sin(2\varphi) \\
&\Gamma_{22,1} = m_{2}R_{1}(R_{2}-r_{2})\sin(\varphi) = K_{3}\sin(\varphi), \quad Q_{2}^{g} = m_{2}g(R_{2}-r_{2})\sin(\varphi) = K_{1}\sin(\varphi)
\end{aligned} \tag{5}$$

For simplicity, we introduce physical parameters $K_{1}, K_{2}, K_{3}, K_{4}$ which are defined as shown above.

3. CONTROLLER DESIGN

In this section a control strategy is developed to stabilize the pendulum in upright position. As mentioned before, there are two different control problems. The first one is swinging the pendulum up from down to the upright position. Once the system is close to the desired position, with a simple change in the controller, it is possible to bring the pendulum in the desired equilibrium.

3.1 Swing up controller

There are many ways to bring the pendulum to the upper half plane, when $|\varphi| < \pi/2$. One of the most popular is based on energy control [3,4]. The goal of this paper is not to build an accurate swing up controller, but to bring the pendulum close enough so the stabilizing controller can stabilize it in the upright position. Hence, the swing up strategy will only be described here in short.

The equation of motion for the pendulum is:

$$\left(J_{\xi 2}+m_2(R_2-r_2)^2\right)\ddot{\varphi}-m_2 g(R_2-r_2)\sin\varphi+m_2 a(R_2-r_2)\cos\varphi=0 \tag{6}$$

where a represents the acceleration of the pendulum's pivot point. Friction has been neglected. For the sake of clarity, let us introduce $J_2=J_{\xi 2}+m_2(R_2-r_2)^2$. The energy of the uncontrolled pendulum (without the rotating arm) is:

$$E=\frac{1}{2}J_2\dot{\varphi}^2+m_2 g(R_2-r_2)(\cos\varphi-1) \tag{7}$$

The energy is defined so that it is zero in upright rest position. Now, it is necessary to understand how the energy is influenced by the acceleration of the pivot. We can find it by computing the time derivative of E:

$$\frac{dE}{dt}=J_2\dot{\varphi}\ddot{\varphi}-m_2 g(R_2-r_2)\dot{\varphi}\sin\varphi=-m_2 a(R_2-r_2)\dot{\varphi}\cos\varphi \tag{8}$$

where Eq. (6) has been used to obtain the last equality. Equation (8) implies that system is simply an integrator with varying gain. To increase energy the acceleration of the pivot a should be positive when the quantity $\dot{\varphi}\cos\varphi$ is negative. With the Lyapunov function $\upsilon=(E_0-E)^2/2$ and the control law $u(t)=a(t)=k(E-E_0)\dot{\varphi}\cos\varphi,\ k=const>0$, it follows:

$$\dot{\upsilon}=-km_2(R_2-r_2)((E_0-E)\dot{\varphi}\cos\varphi)^2 \tag{9}$$

This control law drives the energy towards its desired value $E_0=0$, except when $\dot{\varphi}\cos\varphi=0$.

3.2 Stabilizing controller

Now we can design a controller that stabilizes the pendulum in upright position. For this purpose, we will use nonlinear control technique known as *inverse dynamic control*. It is basically *a partial feedback linearization* procedure [5], which simplifies the control design. The first step of this procedure is to calculate $\ddot{\theta}$ from Eq. (4) and plug it into Eq. (3). After rearranging, Eq. (3) now reads, [6]:

$$\frac{a_{11}}{a_{12}}\left(Q_2^g+\Gamma_{12,1}\dot{\theta}^2\right)+\left(a_{12}-\frac{a_{11}a_{22}}{a_{12}}\right)\ddot{\varphi}+2\Gamma_{12,1}\dot{\theta}\dot{\varphi}+\Gamma_{12,1}\dot{\varphi}^2=M \tag{10}$$

We can see that $\ddot{\theta}$ has been canceled out in (10). Control input M can be chosen as follows:

$$M=\frac{a_{11}}{a_{12}}\left(Q_2^g+\Gamma_{12,1}\dot{\theta}^2\right)+\left(a_{12}-\frac{a_{11}a_{22}}{a_{12}}\right)M_R+2\Gamma_{12,1}\dot{\theta}\dot{\varphi}+\Gamma_{12,1}\dot{\varphi}^2 \tag{11}$$

where M_R is new control input. Now, Eq. (3) and (4) become:

$$\ddot{\theta}=-\frac{K_1}{K_3}\tan(\varphi)-2\frac{K_2}{K_3}\sin(\varphi)\dot{\theta}^2+\frac{K_4}{K_3}\frac{M_R}{\cos(\varphi)} \tag{12}$$

$$\ddot{\varphi}=M_R \tag{13}$$

where physical parameters K_1,K_2,K_3,K_4 are defined in Eqs. (5). Because of the cosine term in term a_{12} in the denominator of Eq. (11), the control signal is defined in every position of the pendulum except for the horizontal, i.e. $|\varphi| < \pi/2$. To achieve asymptotic stability for the $(\varphi, \dot{\varphi})$, a PD controller can be used:

$$M_R = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi} \tag{14}$$

The PD controller stabilizes the inverted pendulum for every $K_{P\varphi}, K_{D\varphi} > 0$, but does not stabilize the arm. This can be seen by observing the zero dynamics of the system. Substituting $\varphi = 0,\ \dot{\varphi} = 0$ into Eq. (12), it follows $\ddot{\theta} = 0 \Rightarrow \dot{\theta} = const$.

So, underactuated mechanical systems like inverted pendulum are not fully feedback linearisable, and control techniques developed for a fully actuated systems cannot be applied here [7]. The new goal is to improve M_R so that asymptotic stability for $(\varphi, \dot{\varphi}, \theta, \dot{\theta})$ can be accomplished. To achieve this, control feedback law will be extended as follows:

$$M_R = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi} - K_{P\theta}\theta\cos(\varphi) - K_{D\theta}\dot{\theta}\cos(\varphi) + \frac{K_1}{K_4}\sin(\varphi) \tag{15}$$

After substituting Eq. (15) into Eq. (12) and (13), we obtain:

$$\ddot{\theta} + \frac{K_4}{K_3}K_{D\theta}\dot{\theta} + \frac{K_4}{K_3}K_{P\theta}\theta = -\frac{K_4}{K_3\cos(\varphi)}\left(K_{D\varphi}\dot{\varphi} + K_{P\varphi}\varphi\right) - 2\frac{K_2}{K_3}\sin(\varphi)\dot{\theta}^2 \tag{16}$$

$$\ddot{\varphi} + K_{D\varphi}\dot{\varphi} + K_{P\varphi}\varphi - \frac{K_1}{K_4}\sin(\varphi) = -\cos(\varphi)\left(K_{D\theta}\dot{\theta} + K_{P\theta}\theta\right) \tag{17}$$

Now, we can linearize system described with Eqs. (16)-(17) around equilibrium point $(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = (0,0,0,0)$. A controller derived from a linearized system will work for a nonlinear system, provided region of attraction is not too large. Under this condition, linearization allows us to neglect nonlinear, quadratic term $\dot{\theta}^2$ in Eq. (16). So, linearization around desired equilibrium point leads to:

$$\ddot{\theta} + \frac{K_4}{K_3}K_{D\theta}\dot{\theta} + \frac{K_4}{K_3}K_{P\theta}\theta = -\frac{K_4}{K_3}K_{D\varphi}\dot{\varphi} - \frac{K_4}{K_3}K_{P\varphi}\varphi \tag{18}$$

$$\ddot{\varphi} + K_{D\varphi}\dot{\varphi} + \left(K_{P\varphi} - \frac{K_1}{K_4}\right)\varphi = -K_{D\theta}\dot{\theta} - K_{P\theta}\theta \tag{19}$$

Choosing the following values for PD parameters:

$$K_{P\theta} = -30;\ K_{D\theta} = -12;\ K_{P\varphi} = 250;\ K_{D\varphi} = 30; \tag{20}$$

where $K_1 = 6.514e-2,\ K_2 = 9.186e-4,\ K_3 = 1.428e-3,$ and $K_4 = 1.837e-3$ are system parameters taken from the real laboratory model of inverted pendulum, eigenvalues of the linearized system are:

$$s_{1,2}^* = -5.6 \pm 9.81j \qquad s_{3,4}^* = -1.67 \pm 2.81j \tag{21}$$

Conditions for asymptotic stability of linearized system are fulfilled. Simulation studies are performed in Matlab Simulink environment to illustrate the performance of the designed controller. Figure 2 below shows results for the change of the pendulum and arm angle, with respect to time. Initial conditions are $(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = (0,0,-\pi,0)$. A change from swing up to stabilizing controller happens when $|\varphi| < \pi/6$.

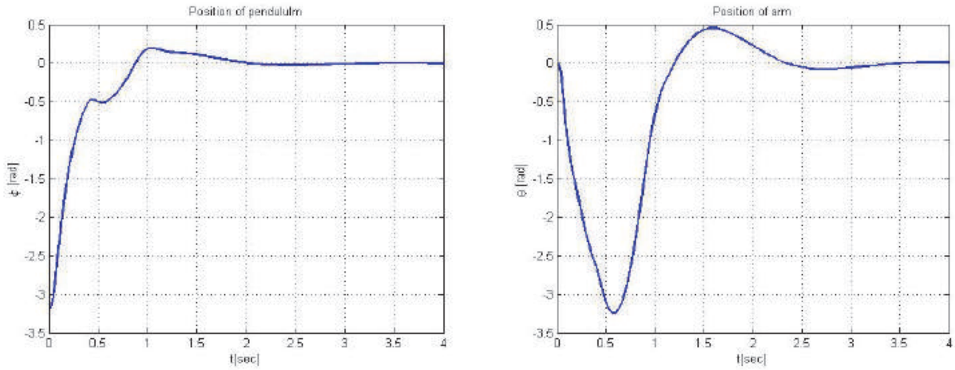

Fig. 2 *Change of pendulum and arm angle*

4. CONCLUSION

In this paper a control algorithm for rotational inverted pendulum is provided. The control strategy consists of two parts, a swing up controller and stabilizing controller. A stabilization algorithm is based on partial feedback linearization, which made it possible to compensate some of nonlinearities of the pendulum. Control feedback law is designed to achieve local asymptotic stability for both the pendulum and the driven arm. Results have been supported by means of the computer simulation. For future research, an improvement of the proposed method is to be considered, based on Lyapunov's direct method. Also, transfer from simulation to real laboratory inverted pendulum will be a subject of future investigations.

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