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THE STATE SPACE MODEL OF A SINGLE-LINK FLEXIBLE ROBOT WITH A FRACTIONAL ORDER VISCOELASTIC ELEMENT IN THE JOINT

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Abstract. This paper extends some results from the area of vibration damping of single-link flexible robots. Instead of the integer derivative model, the fractional order derivative model of a viscoelastic element is proposed. Euler-Bernoulli beam theory for a single-link flexible robot is used. An analytical solution for a set of decoupled ordinary differential equations is found and the state-space model with fractional order derivatives is formed.

1. Introduction

During the last few decades, robotic manipulators have been used to assist in a wide range of tasks. Most of them have been designed with the objective of minimizing the vibration of the end-effector in order to achieve good position accuracy. For rigid link robots it is much easier to ensure positioning accuracy since compliance of such a systems can be neglected. In recent years, the demand for the development of lightweight and flexible manipulators have been increased in the field of space exploration, mining, high precision industrial applications, surgical operations, etc. In case we want to take in to account elastic deformations two cases are apparent: heavy loads and huge operating ranges. But large vibration of the lightweight and flexible manipulator is the major problem over using the conventional manipulator. To minimize the vibration of the manipulator, one may use links with viscoelastic materials for its vibration absorbing behavior. It may be observed that very few works are available on the dynamic analysis of viscoelastic beams. In the literature are usually considered systems with one or two flexible links [1, 2]. For vibration damping of flexible robots active, passive and semi-active methods can be applied [2].

In this paper, one approach in creating the state space model of a single-link flexible robot is given. As a passive damping element in the joint a fractional order viscoelastic model is proposed. Four mode shapes of eigenfrequency function are determined and the dynamic equation of the system with an infinite set of ordinary differential equations is obtained. Finally, state space model of the system with fractional order derivatives is given and some conclusions are made.

2. Modeling of the system

The considered system is a flexible link robot with tipmass driven by a DC motor. We assumed that a robot link is with a homogenous mass distribution and uniform geometric characteristics. Under this assumption the Euler-Bernoulli beam theory for small elastic deformations can be used. If we consider the pinned-free flexible beam with coordinates defined in the Fig.1, with flexural rigidity EI , arm length L and distributed mass μ , then we can obtain a linear model by modal analysis from the following partial differential equation

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right)+\mu\frac{\partial^2 w(x,t)}{\partial t^2}=f(x,t), \tag{1}$$

where $w(x,t)$ is the flexible beam deflection and $f(x,t)$ is the term which includes all external forces. By separating the variables and using the Fourier method we obtain

$$w(x,t)=\sum_{i=0}^{\infty}\phi_i(x)q_i(t), \quad i\in N, \tag{2}$$

where $\phi_i(x)$ are eigen amplitude functions and $q_i(t)$ are time functions.

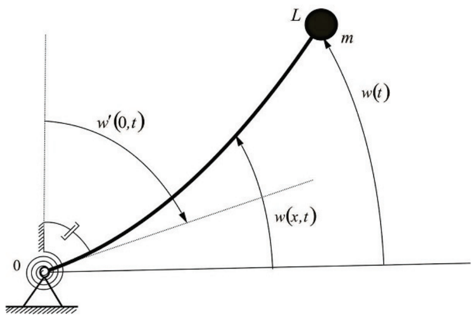

Figure 1. Flexible link with the spring-damper element in the joint

This yields differential equation for the eigenfunctions $\phi(x)$ in the following form

$$\phi^{(IV)}(x)=\lambda^4\phi(x)\ ;\ \lambda^4=\frac{\mu\omega^2}{EI}, \tag{3}$$

with ω denoting the eigenfrequencies. The zeroth mode is the rigid body mode of the link, which characterizes rigid-body manipulators considered without deflection. For $\lambda\neq 0$ solution of fourth order differential equation is in the form

$$\phi_i(x)=A_i\sin(\lambda_i x)+B_i\cos(\lambda_i x)+C_i\sinh(\lambda_i x)+D_i\cosh(\lambda_i x), \tag{4}$$

where after taking in to account following boundary conditions

$$\phi(0)=0;\ \frac{\partial^2}{\partial x^2}\phi(x)\Big|_{x=0}=0;\ \frac{\partial^2}{\partial x^2}\phi(x)\Big|_{x=L}=0;\ \frac{\partial^3}{\partial x^3}\phi(x)\Big|_{x=L}+\frac{m}{\mu}\lambda^4\phi(L)=0, \tag{5}$$

we obtain the following equation

$$\begin{bmatrix}\lambda^2\sin(\lambda L) & -\lambda^2\sinh(\lambda L)\\ -\lambda^3\cos(\lambda L)+\frac{m}{\mu}\lambda^4\sinh(\lambda L) & \lambda^3\cosh(\lambda L)+\frac{m}{\mu}\lambda^4\sinh(\lambda L)\end{bmatrix}\begin{bmatrix}A\\C\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \tag{6}$$

where A and C are integration constants. From Eq. (6) we obtained eigenfrequency equation

$$\sin(\lambda L)\cosh(\lambda L)-\sin(\lambda L)\sinh(\lambda L)+2\frac{m\lambda}{\mu}\sinh(\lambda L)\sin(\lambda L)=0\cdot \tag{7}$$

From the Eq. (7) and by using the software Mathematica eigenvalues of the system are determined for following parameters of the flexible beam: tipmass $m=0.96\,[\mathrm{m}]$, length of the beam $L=0.96\,[\mathrm{m}]$, cross-sectional area of the beam $a=6.08332\times10^{-5}\left[\mathrm{m}^2\right]$ and density of the beam $\rho=2710\left[\mathrm{kg/m}^2\right]$ where $\mu=\rho\cdot a\cdot L$. First four of infinite numbers of eigenvalues are given in the Table 1.

λ_0	λ_1	λ_2	λ_3
0	3.14159	6.28319	9.42478

Table 1. First four eigenvalues of the single link flexible robot structure with tipmass

By using the ratio of cofactors of the Eq. (6) we obtained the eigenfunctions as

$$\phi_i(x)=A_i\left[\sin(\lambda_i x)+\frac{\sin(\lambda_i L)}{\sinh(\lambda_i L)}\sinh(\lambda_i x)\right],\qquad 1\le i\le\infty\ , \tag{8}$$

For so called rigid body mode when we take in to account that $\lambda=0$ we obtain

$$\phi_i(x)=C_0 x\,, \tag{9}$$

where C_0 is the integration constant.

3. State-space representation

Free transverse vibrations of the flexible link are described with the solution (2). We introduce the kinetic energy E_k and potential energy E_p of the system in terms of natural modes as

$$E_k=\frac{1}{2}\delta_{ij}I_b\sum_{i=1}^{n}\dot{q}_i^2\,;\quad E_p=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}q_iq_j\int_0^l EI\phi_i''\phi_j''dx=\frac{1}{2}\delta_{ij}J\sum_{i=1}^{n}\omega_i^2q_i^2 \tag{10}$$

where δ_{ij} is the Kronecker delta symbol, q_i are time dependent generalized coordinates, I_b is inertia moment of the beam, I_p is inertia moment of the tipmass and J is total inertia about the motor armature J. Dissipated energy of the link E_F and the work A of the input torque in the joint are

$$E_F=\frac{1}{2}I_b\sum_{i=1}^{n}2\xi_i\omega_i^2q_i^2\,;\qquad A=Tw'(0,t)=T\sum_{i=1}^{n}\phi_i'(0)q_i\ , \tag{11}$$

Applying the Lagrange approach we obtain dynamic equation of the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)-\frac{\partial L}{\partial q_i}+\frac{\partial E_F}{\partial\dot{q}_i}=A_i \tag{12}$$

where $q_i, \dot{q}_i$ are generalized coordinates and generalized velocities respectively, and A_i represents the work done by the input torque in the joint for each coordinate. If we substitute in Eq. (12) equations for energies and work we obtain an infinite set of decoupled ordinary differential equations as follows

$$\ddot{q}_i(t) + 2\xi_1\omega_1\dot{q}_i(t) + \omega_i^2 q_i(t) = \frac{\phi_i'(0)}{J}T(t), \quad i \in N \ , \tag{13}$$

where $T(t)$ is external torque. In the solution for the transverse vibrations of the flexible beam we depict only first three modes of vibrations and the rigid body mode. Then, the solution can be expressed with the following equation

$$w(x,t) = \sum_{i=0}^{3}\phi_i(x)q_i(t) \tag{14}$$

Here, to reduce the energy of the system we use a spring-damper element (Fig.1). The basic idea is to use the Kelvin-Voigt constitutive relation of a spring-damper element with fractional order derivatives. In a numerous of papers it was shown that fractional derivative models are much more efficient in description of viscoelastic properties than the integer order derivative models. Therefore, it is possible to study the motion of viscoelastically damped structures with a small number of empirical parameters [3, 4]. Here, we used the viscoelastic element with fractional order derivative model as a passive damping element and its influence on the system dynamics can be introduced as a torque in the joint

$$T(t) = -cw'(0,t) - d \quad D^{\alpha}\left(w' \quad (0,t)\right) \tag{15}$$

where $w(0,t)$ and $\dot{w}'(0,t)$ are joint angle and joint angle velocity with the spring constant denoted with c and the damping constant with d. Operator D^{α} denotes the fractional order derivative operator of Riemann-Liouville type [5, 6]. Since spring-damper element is acting as a state feedback

$$T(t) = -c\sum_{i=0}^{3}\phi_0'(0)q_i(t) - d\sum_{i=0}^{3}\phi_0'(0)D^{1/2}\left(q_i(t)\right) = -\mathbf{r}^T\mathbf{x}(t), \tag{16}$$

where $\mathbf{x}(t)$ is the state vector with fractional order $\alpha = 1/2$. The fractional state vector can be defined according to [8] as

$$\mathbf{x}(t) = \begin{bmatrix} q_0 & D^{1/2}q_0 & \dot{q}_0 & D^{3/2}q_0 & q_1 & D^{1/2}q_1 & \dot{q}_1 & D^{3/2}q_1 & q_2 & D^{1/2}q_2 & \dot{q}_2 & D^{3/2}q_2 & q_3 & D^{1/2}q_3 & \dot{q}_3 & D^{3/2}q_3 \end{bmatrix}^T \tag{17}$$

and the static feedback vector is

$$\mathbf{r}^T = \begin{bmatrix} c\phi_0'(0) & d\phi_0'(0) & 0 & 0 & c\phi_1'(0) & d\phi_1'(0) & 0 & 0 & c\phi_2'(0) & d\phi_2'(0) & 0 & 0 & c\phi_3'(0) & d\phi_3'(0) & 0 & 0 \end{bmatrix}^T \tag{18}$$

Then, for the state space model we obtain

$$D^{1/2}\mathbf{x}(t) = (\mathbf{A} - \mathbf{b}\mathbf{r}^T)\mathbf{x}(t) = \mathbf{A}_r\mathbf{x}(t); \quad \mathbf{w}(t) = \mathbf{c}^{\mathrm{T}}\mathbf{x}(t) \ , \tag{19}$$

where the input vector is

$$\mathbf{b}^T = \frac{1}{J}\begin{bmatrix} 0 & \phi_0'(0) & 0 & 0 & 0 & \phi_1'(0) & 0 & 0 & 0 & \phi_2'(0) & 0 & 0 & 0 & \phi_3'(0) & 0 & 0 \end{bmatrix}^T \tag{20}$$

the output vector for the tip position $\mathbf{w}(t)$ is

$$\mathbf{c}^T = \begin{bmatrix} \varphi_0(L) & 0 & 0 & 0 & \varphi_1(L) & 0 & 0 & 0 & \varphi_2(L) & 0 & 0 & 0 & \varphi_3(L) & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

The old system matrix $\mathbf{A}$ is without considered viscoelastic element in the joint and the new system matrix $\mathbf{A}_r$ depending on the properties of the viscoelastic element is

$$\mathbf{A}_r = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_4^* & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_8^* & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
a_{12}^* & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a_{16}^* & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{bmatrix} \tag{22}$$

where are

$$a_4^* = \Bigg[-\frac{\left(\phi_0'(0)\right)^2 c}{J} \quad -\frac{\left(\phi_0'(0)\right)^2 d}{J} \quad 0 \quad 0 \quad -\frac{\phi_0'(0)\phi_1'(0)c}{J} \quad -\frac{\phi_0'(0)\phi_1'(0)d}{J} \quad 0 \quad 0$$

$$-\frac{\phi_0'(0)\phi_2'(0)c}{J} \quad -\frac{\phi_0'(0)\phi_2'(0)d}{J} \quad 0 \quad 0 \quad -\frac{\phi_0'(0)\phi_3'(0)c}{J} \quad -\frac{\phi_0'(0)\phi_3'(0)d}{J} \quad 0 \quad 0 \Bigg]$$

$$a_8^* = \Bigg[-\frac{\phi_1'(0)\phi_0'(0)c}{J} \quad -\frac{\phi_1'(0)\phi_0'(0)d}{J} \quad 0 \quad 0 \quad -\omega_1^2 - \frac{\left(\phi_1'(0)\right)^2 c}{J} \quad -\frac{\left(\phi_1'(0)\right)^2 d}{J} \quad -2\xi_1 \quad 0$$

$$-\frac{\phi_1'(0)\phi_2'(0)c}{J} \quad -\frac{\phi_1'(0)\phi_2'(0)d}{J} \quad 0 \quad 0 \quad -\frac{\phi_1'(0)\phi_3'(0)c}{J} \quad -\frac{\phi_1'(0)\phi_3'(0)d}{J} \quad 0 \quad 0 \Bigg]$$

$$a_{12}^* = \Bigg[-\frac{\phi_2'(0)\phi_0'(0)c}{J} \quad -\frac{\phi_2'(0)\phi_0'(0)d}{J} \quad 0 \quad 0 \quad -\frac{\phi_2'(0)\phi_1'(0)c}{J} \quad -\frac{\phi_2'(0)\phi_1'(0)d}{J} \quad 0 \quad 0$$

$$-\omega_2^2 - \frac{\left(\phi_2'(0)\right)^2 c}{J} \quad -\frac{\left(\phi_2'(0)\right)^2 d}{J} \quad -2\xi_2 \quad 0 \quad -\frac{\phi_2'(0)\phi_3'(0)c}{J} \quad -\frac{\phi_2'(0)\phi_3'(0)d}{J} \quad 0 \quad 0 \Bigg]$$

$$a_{12}^* = \Bigg[-\frac{\phi_3'(0)\phi_0'(0)c}{J} \quad -\frac{\phi_3'(0)\phi_0'(0)d}{J} \quad 0 \quad 0 \quad -\frac{\phi_3'(0)\phi_1'(0)c}{J} \quad -\frac{\phi_3'(0)\phi_1'(0)d}{J} \quad 0 \quad 0$$

$$-\frac{\phi_3'(0)\phi_2'(0)c}{J} \quad -\frac{\phi_3'(0)\phi_2'(0)d}{J} \quad 0 \quad 0 \quad -\omega_3^2 - \frac{\left(\phi_3'(0)\right)^2 c}{J} \quad -\frac{\left(\phi_3'(0)\right)^2 d}{J} \quad -2\xi_3 \quad 0 \Bigg]$$

In that way, the state-space model of the closed-loop system is

$$D^{1/2}x_1 = x_2;\ D^{1/2}x_2 = x_3;\ D^{1/2}x_3 = x_4;\ D^{1/2}x_4 = a_4^* \cdot x;\ D^{1/2}x_5 = x_6;$$

$$D^{1/2}x_6 = x_7;\ D^{1/2}x_7 = x_8;\ D^{1/2}x_8 = a_8^* \cdot x;\ D^{1/2}x_9 = x_{10};$$
$$D^{1/2}x_{10} = x_{11};\ D^{1/2}x_{11} = x_{12};\ D^{1/2}x_{12} = a_{12}^* \cdot x;\ D^{1/2}x_{13} = x_{14};$$
$$D^{1/2}x_{14} = x_{15};\ D^{1/2}x_{15} = x_{16};\ D^{1/2}x_{16} = a_{16}^* \cdot x; \quad (23)$$

For numerical evaluation and simulation, one can obtain equations for the discretized model using definition of the fractional derivatives based on the Grunwald-Letnikov formula [7, 8] where are $D^{\alpha}x(t) = \sum_{j=0}^{[t/T]} b_j^{\alpha} x(t - jT)$, and binomial coefficients $b_o^{\alpha} = 1,\ b_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right) b_{j-1}^{\alpha}$.

After applying the solutions for the state vector Eq. (17) and using the Eq. (19) and Eq. (21) one can obtain a tip position of the flexible link which is the output of our state space model. By changing the spring and damping constants of the viscoelastic element in the joint we can change the system matrix and see its influence on the tip position and damping of the structure vibrations.

4. Conclusion

The idea of using fractional derivative rheological model lies in the fact that such models are known to be very effective in describing the viscoelastic behavior of materials. Also, if applied to dynamic problems the resulting equations of motion can be studied in terms of modal analysis after a fractional state space expansion. It can be stated that the main result of the paper is an efficient method of forming the state space model when a fractional order derivative appears in the dynamic equations. Also, when the proposed damping model is introduced in the flexible link structure the new obtained system matrix depends not only of the properties of the beam but also depends on the properties of viscoelastic element. In the state space model we have possibility to change the fractional order parameter and so in much better way represent the real properties of materials and obtain a required damping. To validate the proposed Grunwald-Letnikov formula for approximation of the fractional derivative it is necessary to perform a numerical experiment. The aim of the future investigation is numerical evaluation of the state space model and to examine an influence of the viscoelastic parameters on the system.

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