A novel ARX-based discretization method for linear non-rational systems

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Abstract

This paper presents a novel, simple, flexible and effective discretization method for linear non-rational systems including arbitrary linear fractional order systems (LFOS). The discretization algorithm relies on the direct integration in the complex domain and application of ARX (*AutoRegressive eXogenous*) model. Parameters of ARX-model are obtained by numerical inversion of Laplace transform from the set of input/output data from recorded step response to model of non-rational system. Numerical simulations of several representatives of LFOS (e.g. fractional order PID controller, fractional logarithmic filter, fractional oscillator etc.) are used to demonstrate the effectiveness of the proposed discretization method, both in the time and frequency domains. The obtained results indicate that the proposed ARX-based discretization method is adequate technique for obtaining digital approximation of LFOS.

Key words: discretization, fractional order systems, AutoRegressive eXogenous model, model reduction, frequency and time domain

1. Introduction

Fractional order systems (FOS) are dynamical systems whose properties are described by fractional-order models, characterized by differential equations with non-integer order differentiation and integrations. Nowadays, FOS are widely used to describe various physical phenomena. Considering fractional derivatives (and integrals) are able to model the non-local and distributed effects, FOS are successfully applicable in natural and technical phenomena for modeling various processes exhibiting memory and/or non-stationary effects. An account of recent application of FC can be found in [1-3], and theoretical aspects are elaborated in [4,5]. Fractional order models have more degree of freedoms than corresponding classical ones which implies that FOS are often more accurate and superior than integer order ones. Additional flexibility in the design process leads naturally to fractional order controllers (FOC). Common example is CRONE (*Commande robuste d'ordre non-entier*, Non-integer robust control) proposed by Oustaloup and coworkers [6,7]. Podlubny [8,9] proposed fractional generalization of PID controller, named as $PI^{\lambda}D^{\mu}$, where both first-order integral and first-order differential actions have been replaced by respective fractional counterparts.

From the theoretical aspect, any FOS/FOC can be seen as a continuous time, linear infinite dimensional filter. Direct implementation of such filter is not possible causing need to find a finite-dimensional approximation of any FOC in general, and fractional differintegrators in particular. Nowadays, modeling, simulation and in the end implementation of the FOC is advanced and facilitated thanks to powerful modern digital computers. Precondition to implement FOC is obtaining discrete equivalent suitable for practical realization of the corresponding control laws, relatively low order with aim to equalize entirely continuous-time system over a wide frequency range. In addition, the fundamental system properties, such are steady-state gain and settling time, as well as basic properties in the frequency domain, must be preserved.

Number of discretization schemes can be principally classified as either direct or indirect. Direct methods are generally based on approximation of the first-order derivative or integral and expansion of non-integer power of this approximation, which is then truncated. Direct method in [10] is based on power series expansion (PSE) of Euler operator, while in [11] Tustin operator is approximated with the continued fraction expansion (CFE). Several another direct discretization schemes are presented in [12-16]. In [17] Barbosa, Tenreiro, Machado, and Silva suggested a novel discretization method based on least-squares fitting in time domain. Indirect methods consists of two transformation stages. First stage comprises derivation a finite dimensional, continuous approximation of the considered FOS. Examples of these approximations are Oustaloup's rational approximation (ORA) reported in [17], sub-optimum H_2 rational approximation was reported in [2]. Simple and an effective approach for non-rational approximation of FOS was elaborated in [18]. Rapajić and al. [19] have been recently proposed efficient and simple, both conceptually and computationally for LFOS applicable to LFOS and arbitrary, linear, stationary, infinite dimensional models. Second stage implies derivation of discrete-time equivalent of previously found continuous-time approximation. Numerous of discretization methods have been proposed in literature to systems described by rational transfer functions: approximation of Euler and Tustin, step-invariant and impulse invariant transformations and others [20]. In [21] Smith proposed a flexible first order, known as T-integrator. Similar tunable first order discretization schemes were proposed by Le Bihan [22], Šekara et al. [23-24]. Efficient implementation of discretization alghorithms was discussed in [25], and both direct and indirect discretization alghorithms were elaborated in [26,27].

The paper is organized as follows. In Section 2, the novel discretization method is elaborated. Section 3 presents several illustrative examples showing effectiveness of the proposed method. Finally, the paper is concluded in Section 4.

2. Discretization method for linear fractional order systems

The discretization method we propose in the current paper consists of three main steps. In the first step, the response of the system to a specific test signal is obtained in the time-domain. The response is evaluated by inversion of the appropriate Laplace transform expressions. The second step is to find an optimal, high-order ARX model describing the obtained response. It is important to note that approaches other than ARX could be used (for example ARMAX, Box-Jenkins, etc.), but for the sake of brevity and without loss of generalization we focus specifically on ARX models in the present work. The third and final step is to perform model reduction in order to obtain a more manageable pulse transfer function. Each of these steps will be elaborated in the remainder of this section. One possible approach to obtain a low-order approximation is to directly use low-order ARX fitting. However, this approach would lead to significant errors even in frequency ranges of interest. Order reduction, however, can be fine-tuned for the specific application at hand, and are in general more flexible. Thus, we chose the three step approach in the current paper.

Step 1. There are a number of criteria for favorable selection of test signals used for process excitation. Among these we emphasize: simple generation and mathematical description, implementation with given actuators, applicability to the process and qualitative excitation of the interesting system dynamics [28]. In order to have signal with zero mean value in the proposed discretization procedure it is used one period of bipolar square wave as a test signal. The selected signal is defined as

$$u(t) = h(t) - 2h(t-\Delta) + h(t+2\Delta) \tag{1}$$

where $h(t)$ is Heaviside function and Δ is time period which is determined on the basis of prior knowledge of the step response of the non-rational transfer function $G(s)$. Laplace transform of the input $u(t)$ is given by

$$U(s) = \frac{(1-e^{-\Delta s})^2}{s} \tag{2}$$

and thus output $y(t)$ can be calculated from inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}\left\{\frac{G(s)(1-e^{-\Delta s})^2}{s}\right\}. \tag{3}$$

Step 2. Measured signals are input $u(t)$ to the LFOS and the output $y(t)$. The input/output data used for estimation of parameters of ARX-model are obtained directly from the recorded step response by means of numerical inversion of Laplace transform. The interested reader is referred to [29]. Discretization of LFOS is based on the estimation of the parameters of a high order ARX model and reduction of the model order to an adequate degree by an appropriate technique. The ARX model structure is usually written in the compact form [30]:

$$y(t)=\frac{B(q)}{A(q)}u(t)+\frac{1}{A(q)}e(t),\quad \begin{array}{l} A(q)=1+a_1z^{-1}+\cdots+a_1z^{-n_\mathrm{A}} \\ B(q)=1+b_1z^{-1}+\cdots+b_1z^{-n_\mathrm{B}} \end{array}. \tag{4}$$

Recorded set of input/output data, i.e. vectors u and y from (3) are incorporated in MATLAB and then used for determination of polynomials $A(q)$ and $B(q)$ from (4), using least-square parameter method according to [31]. It should be emphasized that, in this paper, fixed number of points N is used for determination of inverse Laplace transform in (3), over a selected interval of length T_m. Therefore, the corresponding period of discretization of ARX-model in (4) should be $T_s=T_\mathrm{m}/N$. The overall dynamics of the model is collected by selection of high order degrees n_A and n_B of polynomials $A(q)$ and $B(q)$.

Step 3. In general, the reduction of the model is performed in order to ease the computational efforts of simulation, analysis, design and at the end implementation of the LFOS. Reduction of the obtained ARX model, in this paper, is performed by transformation it into balanced state-space realization according to [32]. Primary idea in this approach is to retain only the most significant states of the model which determine the input-output behavior of the system. Finally, constructed reduced order ARX-model is thus represented by estimated transfer function returned for negative powers of discrete variable z which further can be written in rational transfer function as

$$\hat{G}^{\mathrm{ARX}}_{m/n}(z)=\frac{Y(z)}{U(z)}=\frac{B(z)}{A(z)}=\frac{b_mz^m+b_{m-1}z^{m-1}+\cdots b_0}{a_nz^n+a_{n-1}z^{n-1}+\cdots a_0}. \tag{6}$$

where $m\le n$, and degrees m and n are selected from balanced state-space based ARX-model reduction performed in (6), in order to equalize frequency and time domain characteristics of the LFOS and obtained discrete equivalent. Verification of effectiveness of the proposed method is done by comparison of the time and frequency responses of obtained discrete equivalent and process transfer function $G(s)$.

2. Simulation analysis

The effectiveness of the proposed discretization method is verified via numerical simulations for following transfer functions:

$$G_1(s)=\frac{1}{s^{3/2}+1},\ G_2(s)=\frac{\ln(s)}{s},\ G_3(s)=e^{-\sqrt{s}},\ G_4(s)=\left(\frac{s+1}{0.1s+1}\right)^{0.5},$$

$$G_5(s)=\frac{1+1/s+s^{1.2}}{(0.1s+1)^{1.2}},\ G_6(s)=\frac{1}{s-\sqrt{2s}+1}.$$

Processes with such a transfer function are studied in [26,27,33] which include typical fractional order process $G_1(s)$, fractional logarithmic filter $G_2(s)$, process G_3(s) which is common in analysis of distributed parameter systems, particularly those involving heat and mass transfer; differential compensator $G_4(s)$, fractional order PID controller $G_5(s)$ and fractional oscillator $G_6(s)$. The obtained results are illustrated below in the form of frequency (Bode) and time responses in Fig. 1-6, while coefficients of the discrete equivalents are presented in Table 1.

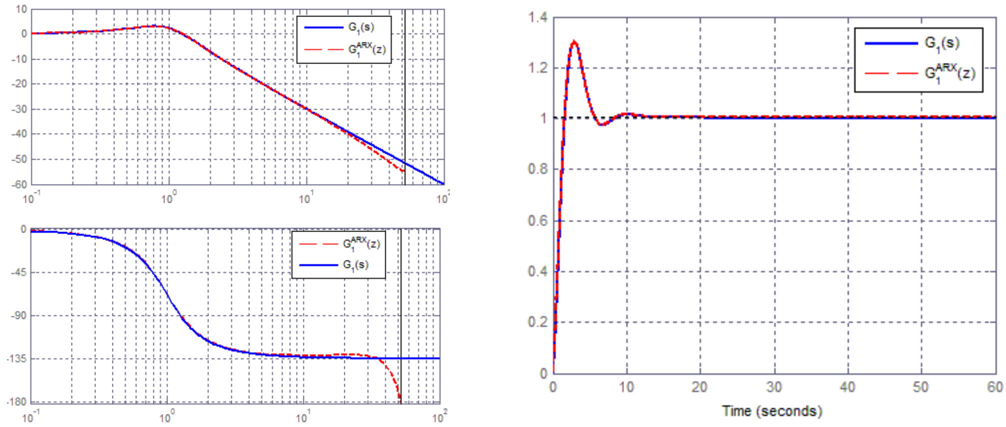

Figure 1. Bode plots (left) and step response (right) for the process $G_1(s)$ (blue) and obtained discrete equivalent $\hat{G}_1^{\mathrm{ARX}}(z)$ (red dashed)

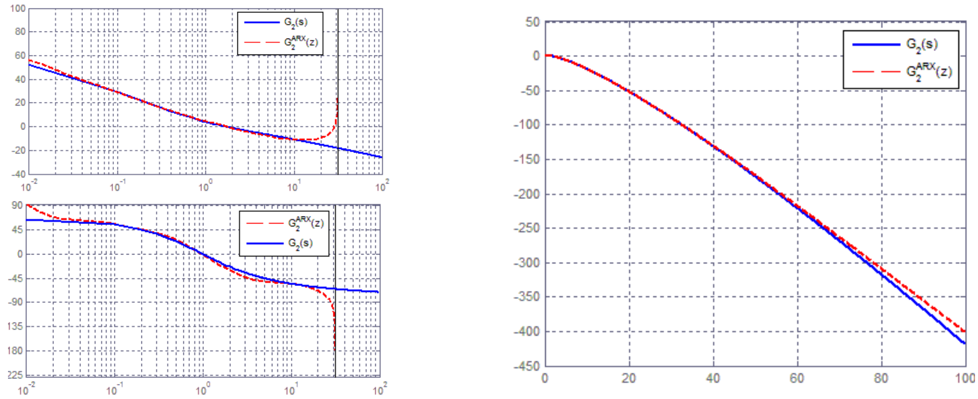

Figure 2. Bode plots (left) and step response (right) for the process $G_2(s)$ (blue) and obtained discrete equivalent $\hat{G}_2^{\mathrm{ARX}}(z)$ (red dashed)

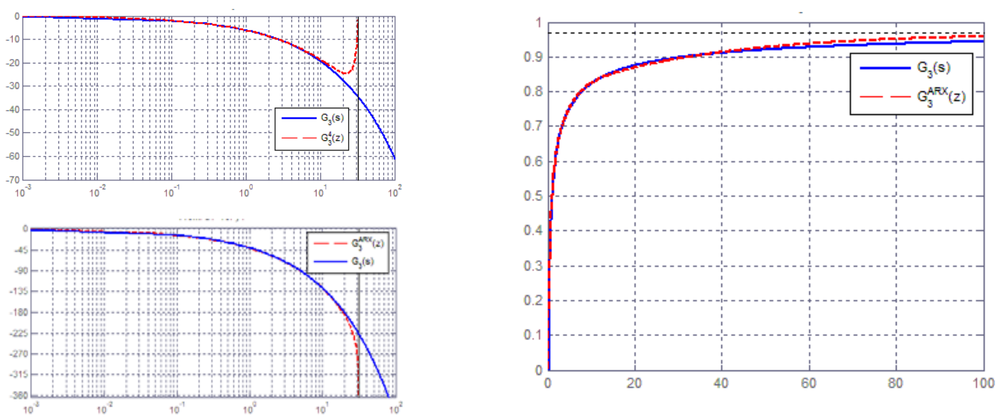

Figure 3. Bode plots (left) and step response (right) for the process $G_3(s)$ (blue) and obtained discrete equivalent $\hat{G}_3^{\mathrm{ARX}}(z)$ (red dashed)

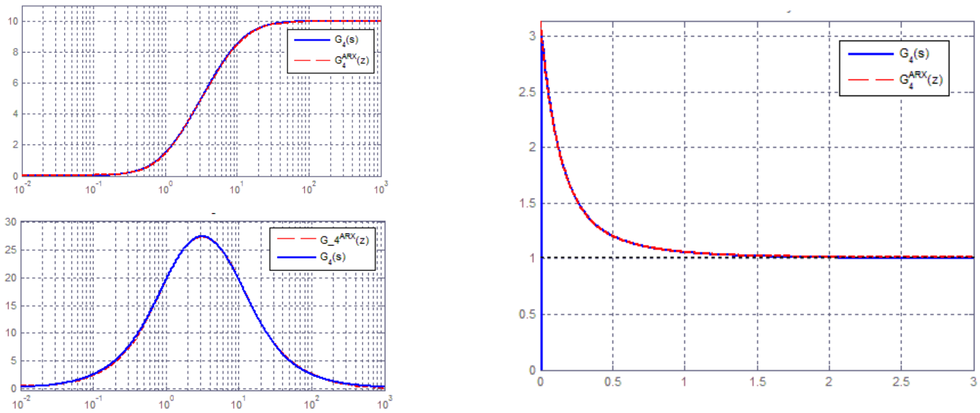

Figure 4. Bode plots (left) and step response (right) for the process $G_4(s)$ (blue) and obtained discrete equivalent $\hat{G}_4^{\mathrm{ARX}}(z)$ (red dashed)

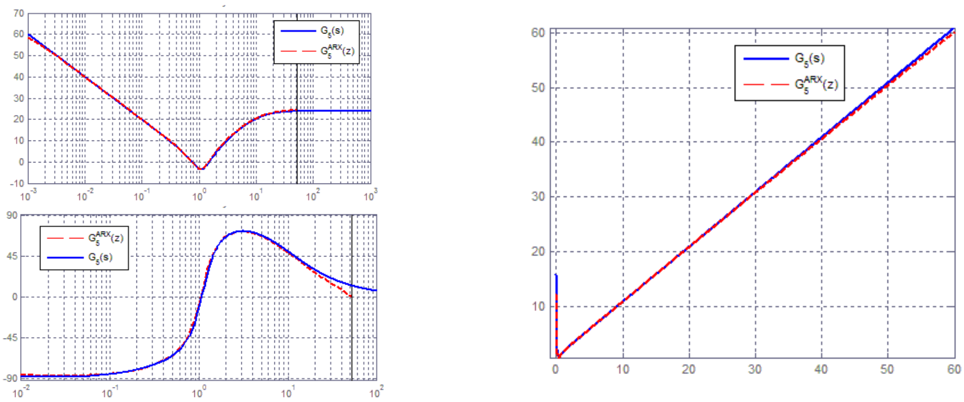

Figure 5. Bode plots (left) and step response (right) for the process $G_5(s)$ (blue) and obtained discrete equivalent $\hat{G}_5^{\mathrm{ARX}}(z)$ (red dashed)

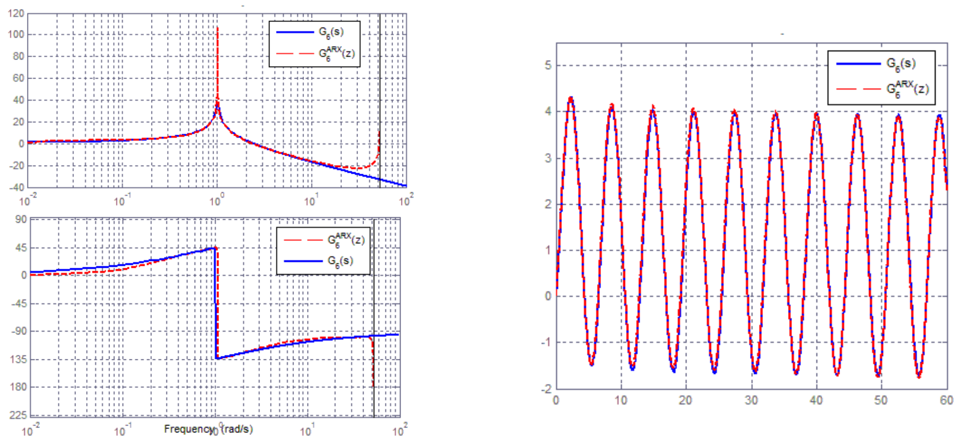

Figure 6. Bode plots (left) and step response (right) for the process $G_6(s)$ (blue) and obtained discrete equivalent $\hat{G}_6^{\mathrm{ARX}}(z)$ (red dashed)

Table 1. Coefficients of transfer functions of the obtained discrete equivalent $\hat{G}^{\text{ARX}}(z)$ for the processes G_j, $j=\overline{1,6}$ for N=1000 and T_s=T_m/N

G_i	Δ [s]	T_m [s]	Coefficients of the $\hat{G}^{\text{ARX}}(z)=B(z)/A(z)$						
			b_6 a_6	b_5 a_5	b_4 a_4	b_3 a_3	b_2 a_2	b_1 a_1	b_0 a_0
G_1	20	60	0.0048 1	-0.0089 -5.3885	-0.0103 12.0357	0.0379 -14.2513	-0.0351 9.4249	0.0130 -3.2964	-0.0016 0.4755
G_2	30	100	0.0349 1	0.3747 -5.3885	-1.1076 12.0357	0.4992 -14.2513	0.8259 9.4249	-0.8045 -3.2964	0.1772 0.4755
G_3	30	100	0.0004 1	0.0605 0.0987	0.1203 -2.6859	-0.1147 -0.1942	-0.2295 2.3929	0.0539 0.0953	0.1090 -0.7066
G_4	1	3	3.318 1	17.2421 -5.4811	39.2919 12.6405	-47.4943 -15.0262	32.0825 10.1271	-11.4674 -3.6118	1.6914 0.5316
G_5	10	60	12.036 1	-47.4483 -3.4287	78.3305 4.9922	-74.6972 -4.3677	48.4545 2.6635	-21.0921 -1.0385	4.4173 0.1792
G_6	18	60	- -	0.0057 1	0.1498 -1.0070	-0.1170 -1.8770	-0.1648 1.9000	0.1057 0.8794	0.0217 -0.8937

As the figures 1-6 show, the proposed method gives adequate discrete equivalents for a wide range of LFOS. As it can be seen from Table 1, obtained transfer functions G_1-G_5 are of 6th order, and for G_6 of 5th order. Compared to recently proposed discretization methods, reported in [18,19], where 7th order is obtained, the presented method is more flexible with easier construction of lower model order and preservation of continuous-time characteristics in discrete-time domain. Obtained frequency characteristics for all examples of LFOS considered here, gives adequate results and can be easily processed by modern computers. It should be mentioned that all of the time-domain responses presented above are obtained by means of direct integration in the complex domain. For more information we refer to [29]. Besides, it should be pointed out that in this paper for discretization procedure the fixed number N=1000 has been chosen in order to decrease the computation error of inverse Laplace transform. However, another value of N could be used, by taking into account that sampling period T_s=T_m/N is than indirectly defined on the basis of selected values of T_m and N.

It is important to note that there is a number of methods for rationalization and/or discretization of fractional and non-rational transfer functions. As it can be seen from frequency characteristics is that relative error percentage over a wide frequency range is less than 1%, while maximum relative error is corresponding to the Nyquist frequency. Full comparison with available methods with error bound computation of each step approximation is beyond the scope of the present paper and will be the subject of future research. An example, of how the approximation error can be considered to create uncertain order model is recently reported in [34]. However, it is worth stressing that the proposed method is more flexible and precise than most of other recently reported ones.

3. Conclusions

The method proposed here is based on ARX-based parameter estimation of the model of the fractional order process previously excited by double rectangular pulse. The parameters of ARX-model are obtained by means of numerical inversion of Laplace transform on the basis of the input/output data from recorded step response to the examined fractional order model. Obtained results presented here indicate that proposed ARX-based discretization method is adequate technique for discretizing LFOS. The method is characterized with computational efficiency, flexibility and effectiveness, as it is illustrated by numerous examples.

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