Stabilization of the cart pendulum system by fractional order control with experimental realization

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Abstract

This paper deals with stability problem of cart inverted pendulum system controlled by a fractional order controller. Inverted pendulum is an underactuated mechanical system with one control input and two degrees of freedom. Detailed mathematical model of pendulum is derived using the Rodriguez method. Stabilization of pendulum around its unstable equilibrium point is achieved by using the fractional order PD^{α} controller, in combination with partial feedback linearization technique. Since fractional order control law includes non-rational functions, an efficient method for numerical evaluation of this type of functions is used in this paper. The performance of the proposed method is demonstrated with experimental verification of the stabilization control of the cart pendulum system.

Key words: inverted pendulum, fractional order PID control, asymptotic stability, rational approximations

1. Introduction

Many systems in nature are inherently underactuated, with fewer actuators than the number of degrees of freedom. These systems have been widely studied in diverse fields, such as robotics, aerospace engineering, marine engineering etc. Classical benchmark examples of underactuated mechanical systems used in control theory are inverted pendulums among which, the cart pendulum [1], the Furuta pendulum [2], the Acrobot [3], and the Pendubot [4] are very popular. A vast range of different nonlinear control algorithms exists for the stabilization of these types of pendulums, including feedback linearization methods [5], a combined feedforward/feedback control schemes [6], variable structure controls [7], etc. Some results of the D-decomposition procedure for underactuated systems have been given in [8,9,10]. Unlike fully actuated systems, here feedback linearization technique cannot be applied directly, because of the unstable zero dynamics. To solve this problem, additional control law must be designed to guarantee asymptotic stability of the system.

On the other hand, in recent years considerable attention has been paid to fractional calculus and its application [11]. It is a mathematical topic with more than 300 years old history, but its application to physics and engineering has been recorded only in the past few decades. The fractional calculus is a generalization of classical, integer order calculus [12], and has the potential to accomplish what classical integro-differential operators cannot.

In control theory fractional order controllers are used to improve the performance of closed loop systems. The fractional PID controllers, the CRONE controllers, the fractional lead-lag compensators etc., are some of the well known controllers of fractional type. Among them, fractional order PID controllers are the ones most frequently used and were first introduced in [13]. It has been shown that fractional order PID controller enhances the system control performances when used with integer order and fractional order plants. In this paper, fractional order PD control is used for stabilization of cart pendulum system, after the partial feedback linearization procedure has been applied in order to simplify the control design problem.

Linear fractional order control laws are represented by a transfer functions which are not rational, which gives rise to a problem of practical implementation of the corresponding control algorithms. A method for rational approximation of linear fractional order systems used in this paper is computationally efficient, accurate, and has originally been proposed in [14]. It relies on the interpolation of the frequency characteristics of the system on a predefined set of target frequencies.

The rest of the paper is organized as follows. First, mathematical model of cart pendulum system is presented. Then, a fractional order PD^{α} controller in combination with partial feedback linearization technique is introduced in order to stabilize the system. Experimental results of the proposed algorithm for the stabilization of cart pendulum system are given at the end to demonstrate the validity of the presented method.

2. Dynamics of cart pendulum system (medium pendulum case)

In Figure 1 a laboratory setup of cart pendulum system is shown, while Fig. 2 depicts a schematic view of the same. It is a mechanical system with two degrees of freedom, where the cart position and the pendulum angle are denoted as x and φ, respectively. Control of the system is by means of force F applied horizontally to the cart. Hence, it is an underactuated mechanical system because it has only one control input and two degrees of freedom.

Table 1. Physical parameters of laboratory cart pendulum system

$M = 0.75$ kg	mass of the cart
$m = 0.127$ kg	mass of the medium pendulum
$g = 9.81 \mathrm{m/s^2}$	gravitational constant

$L = 0.3365$ m	total length of the pendulum
$d = 0.1778$ m	distance of the pendulum's pivot point to its center of mass
$J = 3.1613 \cdot 10^{-2}$ kgm^2	moment of inertia of the pendulum with respect to its pivot point
$\beta_C = 4$ Ns/m	viscous damping coefficient (cart)
$\beta_P = 0.003$ Ns	viscous damping coefficient (pendulum)

Figure 1. A laboratory setup of cart pendulum system

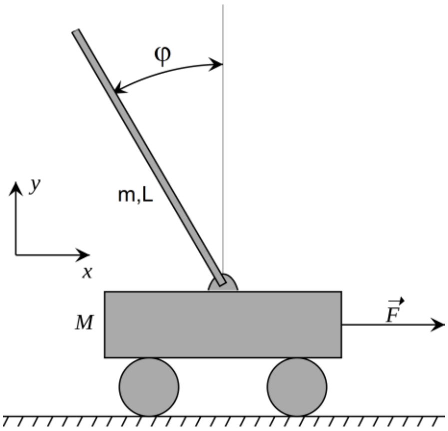

Figure 2. A schematic view of cart pendulum system

The physical parameters of the actual system used in experiment are given in Table 1. Herein, the Rodriguez method [15] is proposed for modeling the dynamics of the system where configuration of the mechanical model can be defined by generalized coordinates q_1 and q_2 represent by x and φ, respectively. The equations of motion of the inverted pendulum can be expressed in a covariant form of Langrange's equation of second kind as follows

$$\sum_{\alpha=1}^{n} a_{\gamma\alpha}\ddot{q}_{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \Gamma_{\alpha\beta,\gamma}\dot{q}_{\alpha}\dot{q}_{\beta} = Q_{\gamma}, \quad \gamma = 1,2\,, \tag{1}$$

wherein the coefficients $a_{\alpha\beta}$ are the covariant coordinates of the basic metric tensor $[a_{\gamma\alpha}] \in R^{2\times 2}$ and $\Gamma_{\alpha\beta,\gamma}$ $\alpha,\beta,\gamma = 1,2$ presents Christoffel symbols of the first kind. The generalized forces Q_γ can be presented in the following expression (2), wherein Q_γ^g, Q_γ^v, Q_γ^a denote the generalized gravitational, viscous and control forces, respectively.

$$Q_\gamma = Q_\gamma^g + Q_\gamma^v + Q_\gamma^a, \quad \gamma = 1,2\,. \tag{2}$$

The equations of motion of our system can be rewritten in full form

$$(m+M)\ddot{x} - md\ddot{\varphi}\cos(\varphi) + md\dot{\varphi}^2\sin(\varphi) = F - \beta_C\dot{x}\,, \tag{3}$$

$$-md\cos(\varphi)\ddot{x} + J\ddot{\varphi} = mgd\sin(\varphi) - \beta_P\dot{\varphi}\,, \tag{4}$$

wherein

$$a_{11} = m+M,\ a_{12} = -md\cos(\varphi),\ \Gamma_{22,1} = md\sin(\varphi),\ Q_1^a = F,\ Q_2^v = -\beta_C\dot{x}, \tag{5}$$

$$a_{21} = -md\cos(\varphi),\ a_{22} = J,\ Q_2^g = mgd\sin(\varphi),\ Q_2^v = -\beta_P\dot{\varphi}. \tag{6}$$

3. Controller design of cart pendulum system

Now, a control strategy is developed to stabilize the pendulum in its unstable upright position. First, we show the simplification of dynamic equations of the cart pendulum system. For this purpose, we use nonlinear control technique known as inverse dynamic control. It is basically a partial feedback linearization procedure [16], which simplifies the control design. The first step of this procedure is to calculate $\ddot{\varphi}$ from Eq. (4) and plug it into Eq. (3). After rearranging, Eq. (3) now reads:

$$\left(m+M-\frac{m^2d^2}{J}\cos^2(\varphi)\right)\ddot{x} + \beta_P\dot{x} - \frac{m^2d^2g}{J}\sin(\varphi)\cos(\varphi) - md\dot{\varphi}^2\sin(\varphi) - \frac{md\beta_P}{J}\dot{\varphi}\cos(\varphi) = F. \tag{7}$$

We can see that $\ddot{\varphi}$ has been canceled out in (7). Control force F can be chosen as follows:

$$\left(m+M-\frac{m^2d^2}{J}\cos^2(\varphi)\right)F_R + \beta_P\dot{x} - \frac{m^2d^2g}{J}\sin(\varphi)\cos(\varphi) - md\dot{\varphi}^2\sin(\varphi) - \frac{md\beta_P}{J}\dot{\varphi}\cos(\varphi) = F, \tag{8}$$

where F_R is new control signal. Now, Eqs. (3)-(4) become

$$\ddot{x} = F_R\,, \tag{9}$$

$$J\ddot{\varphi} - mgd\sin(\varphi) + \beta_P\dot{\varphi} = mdF_R\cos(\varphi)\,. \tag{10}$$

We can see there is no influence from the motion of pendulum to cart position in these equations. Now, we can linearize the system described with (9)-(10) around equilibrium point $(x,\dot{x},\varphi,\dot{\varphi}) = (0,0,0,0)$. A controller derived from a linearized system will work for a nonlinear system, provided that region of attraction is not too large [16]. So, linearization around desired equilibrium point leads to:

$$\ddot{x} = F_R \tag{11}$$

$$J\ddot{\varphi} - mgd\varphi + \beta_P\dot{\varphi} = mdF_R \tag{12}$$

The new goal is to choose M_R so that asymptotic stability for $(x, \dot{x}, \varphi, \dot{\varphi}) = (0,0,0,0)$ can be accomplished. This can be achieved with the following control law:

$$F_R = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi} - K_{Dx}x^{(\alpha)} - K_{Px}x\,, \tag{13}$$

wherein $K_{Px}, K_{Dx}, K_{P\varphi}, K_{D\varphi}$ denote proportional and differential gains of the controller, and α is real differentiator parameter. After substituting (13) into (11)-(12), we obtain:

$$\ddot{x} + K_{Dx}x^{(\alpha)} + K_{Px}x = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi}, \tag{14}$$

$$J\ddot{\varphi} + \left(mdK_{D\varphi} + \beta_P\right)\dot{\varphi} + md\left(K_{P\varphi} - g\right)\varphi = -mdK_{Dx}x^{(\alpha)} - mdK_{Px}x \tag{15}$$

Taking $\alpha = 1$ we obtain classical PD controller. Five parameters $\left(K_{Px}, K_{Dx}, K_{P\varphi}, K_{D\varphi}, \alpha\right)$ can be changed in order to achieve asymptotic or relative stability of closed loop system.

4. Experimental results

To show the practical implementation of the fractional order PD controller, we performed experiments for the asymptotic stabilization of the cart pendulum system given by (3)-(4). Some preliminary results regarding the fractional order PD control on cart pendulum system (long pendulum case) is given in [17]. The physical parameters of the actual system are specified in Section 2. The cart is driven by the force generated by the DC motor given by:

$$F = c_m V_m - c_b\dot{x}\,, \tag{16}$$

where V_m is a control input voltage, and $c_m = 1.0717\left[\mathrm{N/V}\right]$ and $c_b = 4.809\left[\mathrm{Ns/m}\right]$ are positive motor constants. The cart pendulum system used in experiments is equipped with two incremental encoders for the measurement of both positions of the cart and pendulum. Since numerical differentiation usually introduces significant noise in velocity measurements, we estimate the corresponding velocities of the cart and pendulum from position measurements by utilizing a derivative filter given by the following fractional order transfer function:

$$DF(s) = \left(\frac{s}{0.02s + 1}\right)^{\alpha}, \tag{17}$$

wherein α represents real differentiator parameter, having in mind that $\alpha = 1$ for pendulums derivative filter. In order to approximate this non-rational transfer function, a computationally efficient method for rational approximation of linear fractional order systems is used, as described in [14]. Rational transfer function of the following form:

$$\frac{B(s)}{A(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{18}$$

should approximate transfer function given by (17). For $b_0 = 1$, there are $2n$ unknown coefficients $(a_0, a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_n)$ which should be determined from $2n$ equations obtained from the condition of overlapping the Bode frequency characteristics in the predefined set of target frequencies $\omega \in [\omega_0, \omega_1, \ldots, \omega_{n-1}]$. For more information we refer to [14]. In our case, for the given non-rational transfer function (17), and for the following values of parameter $\alpha \in [0.9,\ 0.95,\ 1.05,\ 1.1,\ 1.15]$, rational approximation (18) is obtained by interpolating frequency response in seven target points $\omega \in [1,\ 10,\ 20,\ 40,\ 70,\ 100,\ 1000]$. For these target frequencies unknown coefficients in (18) are calculated and listed in Table 2.

Table 2. Coefficients of the rational transfer function

Results	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 1.05$	$\alpha = 1.1$	$\alpha = 1.15$
b_0	1	1	-1	-1	-1
a_0	25.24	53.2	60.24	32.19	22.96
b_1	28.35	56.45	56.60	28.32	18.86
a_1	32.66	66.24	70.46	36.52	25.22
b_2	30.09	63.06	71.54	38.24	27.23
a_2	9.92	19.55	19.88	10.08	6.8
b_3	7.75	16.68	20.22	11.19	8.235
a_3	1.10	2.12	2.05	1.04	0.689
b_4	0. 705	1.54	1.96	1.11	0.834
a_4	0.529e-1	0.100	0.955e-1	0.47e-1	0.307e-1
b_5	0.248	0.55e-1	0.723e-1	0.418e-1	0.319e-1
a_5	0.112e-2	0.208e-2	0.194e-2	0.947e-3	0.61e-3
b_6	0.289e-3	0.647e-3	0.875e-3	0.513e-3	0.397e-3
a_6	0.856e-5	0.157e-4	0.144e-4	0.694e-5	0.442e-5
a_7	0	0	0	0	0

Fig. 3 and 4 show experimental results of the asymptotic stabilization of the laboratory cart pendulum system for different values of α, i.e. $\alpha \in [0.9;\ 0.95;\ 1.0;\ 1.05\ ;\ 1.1;\ 1.15]$. For better visibility results are shown on two pictures instead of one. Tests are performed for the following values of controller parameters: $K_{Px} = 14.14$, $K_{Dx} = 15.58$, $K_{P\varphi} = 66.26$, and $K_{D\varphi} = 10.7$. Pendulum is manually brought to the initial position $(x, \dot{x}, \varphi, \dot{\varphi}) = (0, 0, 165°, 0)$ where balancing controller catches it and stabilizes. We can see the best response is obtained for $\alpha = 1.1$, so fractional order controller gives better control performances when compared with its integer order counterpart.

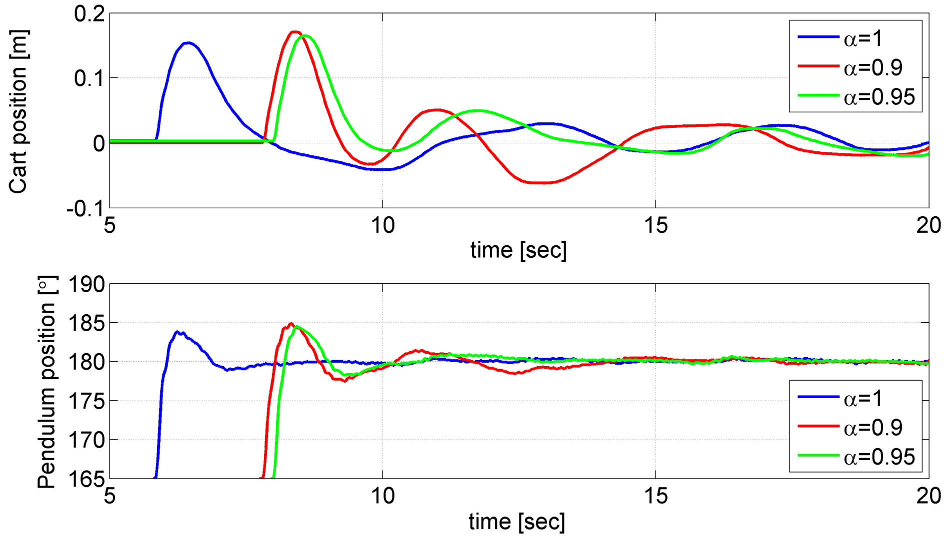

Figure 3. Cart position and pendulum angle for $\alpha \in [0.9;\ 0.95;\ 1.0]$

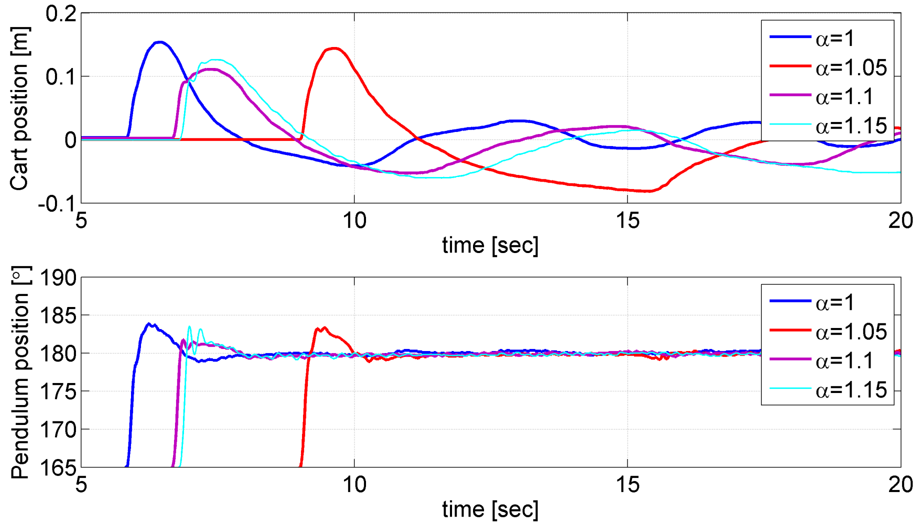

Figure 4. Cart position and pendulum angle for $\alpha \in [1.0;\ 1.05\ ;\ 1.1;\ 1.15]$

5. Conclusions

In this paper, stability problem of cart inverted pendulum system controlled by a fractional order controller is investigated. Detailed mathematical model of pendulum is derived using the Rodriguez method. Stabilization of pendulum around its unstable equilibrium point is achieved by using the fractional order PD$^{\alpha}$ controller, in combination with partial feedback linearization technique. An efficient method for numerical evaluation of non-rational functions is used in order to implement fractional order control law. The performance of the proposed method is demonstrated with experimental verification of the stabilization control of the cart pendulum system. It has been shown that fractional order PD controller can enhance the system control performances in comparison with classical, integer order controller.

Acknowledgments

Authors gratefully acknowledge the support of Ministry of Education, Science and Technological Development of the Republic of Serbia under the project TR 33047 (P.D.M.) as well as supported by projects TR 35006 (M.P.L.) and TR 33020 (T.B.Š).

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