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CONTROL OF THE CART PENDULUM SYSTEM BY USING A FRACTIONAL ORDER PD CONTROLLER

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Summary: *This paper deals with stability problem of cart inverted pendulum controlled by a fractional order PD controller. Inverted pendulum is an underactuated mechanical system because it has one control input and two degrees of freedom. Mathematical model of cart pendulum system is derived and fractional order PD controller is introduced in order to stabilize it. Control strategy consists of two parts, a swing up controller and stabilizing controller. Problem of asymptotic stability of closed loop system is solved using the D-decomposition approach. Stability regions in control parameters space are calculated using this method, and tuning of the fractional order controller can be carried out.*

Key words: inverted pendulum, fractional order PID, D-decomposition, asymptotic stability

1. INTRODUCTION

Inverted pendulum is one of the most interesting problems in control theory and has been studied through many researches in control community. It is nonlinear, unstable and underactuated system, and thus an excellent benchmark for testing different control algorithms. On the other hand, in recent years considerable attention has been paid to fractional calculus and its application. In control theory fractional order controllers are used to improve the performance of closed loop systems. Among them, fractional order PID controllers are the ones most frequently used and were first introduced in. It has been shown that fractional order PID controller enhances the system control performance when used with integer order and fractional order plants.

One of the basic requirements in control systems is their asymptotic stability. There are several methods for determining stability region of a closed loop system, and D-decomposition is one of them. In this paper, D-decomposition method is applied to the inverted pendulum case, and determining its stability regions in parameters space of a fractional order PD controller is presented. D-decomposition for linear fractional systems is investigated, and for the case of linear parameters dependence. This technique enables efficient computational method for determining the asymptotic stability region. When stability regions are known, tuning of the fractional order controller can be carried out.

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First, mathematical model of cart pendulum system is presented. Then, a fractional order PD controller is introduced in order to stabilize the pendulum. Method for tuning the parameters of fractional order controller is given, using the abovementioned D-decomposition method. At the end, example is given and tests are made in order to verify that stability domains are well calculated.

2. DYNAMIC EQUATIONS OF CART PENDULUM SYSTEM

In Fig. 1 a schematic of cart pendulum system is shown. It is a mechanical system with two degrees of freedom, where the cart position and the pendulum angle are denoted as x and φ, respectively. Control of the system is by means of force F applied horizontally to the cart. Hence, it is an underactuated mechanical system because it has only one control input and two degrees of freedom.

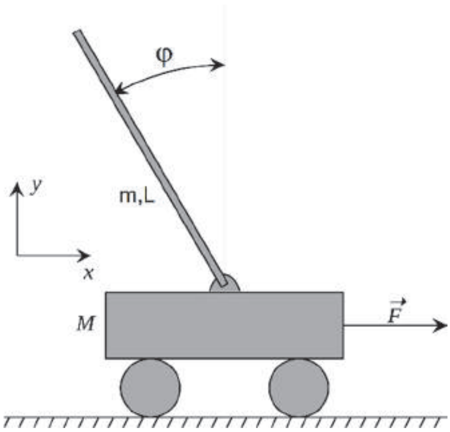

Fig. 1 C*art pendulum system*

Parameters of the system are: M - mass of the cart, m - mass of the pendulum, L - total length of the pendulum, $0.5L$ - distance of the pendulum's pivot point to its center of mass, J - moment of inertia of the pendulum with respect to its pivot point. Applying Lagrange's method [1], the system's nonlinear equations of motion can be readily obtained, and can be expressed in the form

$$(m+M)\ddot{x}+\frac{mL}{2}\left(-\ddot{\varphi}\cos(\varphi)+\dot{\varphi}^2\sin(\varphi)\right)=F\,, \qquad \frac{m}{2}\ddot{x}\cos(\varphi)-\frac{mL}{3}\ddot{\varphi}+\frac{mg}{2}\sin(\varphi)=0\,. \quad (1)\text{-}(2)$$

Now, we show the simplification of dynamic equations of the cart pendulum system. For this purpose, we use nonlinear control technique known as inverse dynamic control. It is basically a partial feedback linearization procedure [2], which simplifies the control design. The first step of this procedure is to calculate $\ddot{\varphi}$ from Eq. (2) and plug it into Eq. (1). After rearranging, Eq. (1) now reads

$$\left(m+M-\frac{3m}{4}\cos^2(\varphi)\right)\ddot{x}-\frac{3mg}{4}\sin(\varphi)\cos(\varphi)+\frac{mL}{2}\dot{\varphi}^2\sin(\varphi)=F\,. \quad (3)$$

We can see that $\ddot{\varphi}$ has been canceled out in (3). Control force F can be chosen as follows

$$F = \left(m + M - \frac{3m}{4}\cos^2(\varphi) \right) F_R - \frac{3mg}{4}\sin(\varphi)\cos(\varphi) + \frac{mL}{2}\dot{\varphi}^2 \sin(\varphi), \tag{4}$$

where F_R is new control signal. Now, Eqs. (1)-(2) become

$$\ddot{x} = F_R, \qquad \ddot{\varphi} = \frac{g}{b}\sin(\varphi) + \frac{1}{b} F_R \cos(\varphi). \tag{5)-(6}$$

wherein $g \approx 9.81\left[m/s^2 \right]$ and $b = 2L/3$. We can see there is no influence from the motion of pendulum to cart position in this equations.

3. CONTROLLER DESIGN

Now, a control strategy is developed to stabilize the pendulum in upright position, and it consists of two different control problems. The first one is swinging the pendulum up from down to the upright position. Once the system is close to the desired position, with a simple change in the controller, it is possible to bring the pendulum in the desired equilibrium.

3.1 Swing up controller

There are many ways to bring the pendulum to the upper half plane, when $|\varphi| < \pi/2$. One of the most popular is based on energy control [3]. The equation of motion for the pendulum is given by Eq. (6). The energy of the uncontrolled pendulum ($F_R = 0$) is

$$E = \frac{1}{2} J \dot{\varphi}^2 + mg\frac{L}{2}\left(\cos(\varphi) + 1 \right). \tag{7}$$

The energy is defined so that it is zero in downright rest position. Now, it is necessary to understand how the energy is influenced by the control input F_R. We can find it by computing the time derivative of E

$$\frac{dE}{dt} = J\dot{\varphi}\ddot{\varphi} - mg\frac{L}{2}\dot{\varphi}\sin(\varphi) = \frac{J}{b}\dot{\varphi}\cos(\varphi) F_R, \tag{8}$$

where Eq. (6) has been used to obtain the last equality. Equation (8) implies that system is simply an integrator with varying gain. To increase energy the control signal F_R should be positive when the quantity $\dot{\varphi}\cos(\varphi)$ is negative. With the Lyapunov function

$$V = \frac{(E - E_0)^2}{2}, \tag{9}$$

and the control law

$$F_R = -k(E - E_0)\dot{\varphi}\cos(\varphi), \; k = const > 0, \tag{10}$$

we find that

$$\dot{V} = \frac{dV}{dt} = -k\frac{J}{b}\left((E - E_0)\dot{\varphi}\cos(\varphi) \right)^2. \tag{11}$$

This control law drives the energy towards its desired value $E_0 = mgL$ (inverted vertical position of pendulum), except when $\dot{\varphi}\cos\varphi = 0$.

3.2 Stabilizing controller

Equations (5)-(6) are used for computer simulation of the system, but for controller design we must linearise Eq. (6) around unstable equilibrium point $(\dot{x},\varphi,\dot{\varphi})=(0,0,0)$. A controller derived from a linearized system will work for a nonlinear system, provided that region of attraction is not too large [2]. So, linearization around desired equilibrium point leads to

$$\ddot{x}=F_R\ , \qquad \ddot{\varphi}=\frac{g}{b}\varphi+\frac{1}{b}F_R\,. \tag{12)-(13}$$

We can see Eq. (5) is already linear, so it remains the same. New goal is to design F_R so that asymptotic stability for $(\dot{x},\varphi,\dot{\varphi})$ can be accomplished. To achieve this, an extended fractional order PD controller is proposed, as a generalization of the PID controller [4]. Control feedback law will be extended as follows

$$F_R=-K_{P\varphi}\varphi-K_{D\varphi}\varphi^{(\beta)}-K_{Dx}\dot{x}\ , \tag{14}$$

wherein $K_{P\varphi},K_{D\varphi},K_{Dx}$ denote proportional and differential gains of the controller, and β is real differentiator parameter. After substituting Eq. (14) into Eqs. (12)-(13), we obtain

$$\ddot{x}+K_{Dx}\dot{x}=-K_{P\varphi}\varphi-K_{D\varphi}\varphi^{(\beta)}\ , \qquad b\ddot{\varphi}+K_{D\varphi}\varphi^{(\beta)}+\left(K_{P\varphi}-g\right)\varphi=-K_{Dx}\dot{x}\ . \tag{15)-(16}$$

Taking and $\beta=1$ we obtain classical PD controller. Four parameters $\left(K_{Dx},K_{P\varphi},K_{D\varphi},\beta\right)$ in Eq. (14) can be changed in order to achieve asymptotic or relative stability of closed loop system. Goal of this paper is to determine the influence of $K_{P\varphi}$, $K_{D\varphi}$ and β parameters on asymptotic stability of system described with Eqs.(15)-(16).

4. D-DECOMPOSITION METHOD

Using the classical D-decomposition method [5] the stability region in the parameter plane $\left(K_{P\varphi},K_{D\varphi}\right)$ may be determined. The characteristic polynomial of the closed loop system described with (15)-(16) is given by:

$$f(s)=b\,s^3+bK_{Dx}s^2+K_{D\varphi}s^{1+\beta}+\left(K_{P\varphi}-g\right)s-gK_{Dx} \tag{17}$$

Plane $\left(K_{P\varphi},K_{D\varphi}\right)$ is decomposed by the boundaries of the D-decomposition into finite number regions D(*k*). Any point in D(*k*) corresponds to such values of $K_{P\varphi}$ and $K_{D\varphi}$ that polynomial (17) has exactly *k* zeroes with positive real parts. The region D(*0*) represents the stability region. Stability boundaries are curves on which each point corresponds to polynomial (17) having zeroes on the imaginary axes. We obtain this boundary by substituting $s=j\omega$ in Eq. (17) and equating it to 0, i.e.

$$b\left(j\omega\right)^3+bK_{Dx}(j\omega)^2+K_{D\varphi}(j\omega)^{1+\beta}+\left(K_{P\varphi}-g\right)(j\omega)-gK_{Dx}=0\,. \tag{18}$$

Term $(j\omega)^{\beta+1}$ in equation above can be expressed as

$$(j\omega)^{\beta+1}=\omega^{\beta+1}\left(\cos(\beta+1)\pi/2+j\sin(\beta+1)\pi/2\right),\ \ \omega\geq 0 \tag{19}$$

Complex equation (18) can be rewritten as:

$$u\left(\omega, K_{P\varphi}, K_{D\varphi}, \beta\right) + jv\left(\omega, K_{P\varphi}, K_{D\varphi}, \beta\right) = 0 \quad (20)$$

where $u\left(\omega, K_{P\varphi}, K_{D\varphi}, \beta\right)$ and $v\left(\omega, K_{P\varphi}, K_{D\varphi}, \beta\right)$ denote the real and imaginary part of (18). Equating the real and imaginary part of Eq. (20) to zero, and solving it for parameters $\left(K_{P\varphi}, K_{D\varphi}\right)$, we obtain

$$K_{D\varphi} = \frac{a + b\omega^2}{\omega^{1+\beta}\cos\left(\beta+1\right)\pi/2} K_{Dx}, \qquad K_{P\varphi} = a + b\omega^2 - K_{D\varphi}\omega^{\beta}\sin\left(\beta+1\right)\pi/2. \quad (21)\text{-}(22)$$

Equations (21) and (22) determine the stability boundaries in parameter space $\left(K_{P\varphi}, K_{D\varphi}\right)$ for $\omega \in \left(0, \infty\right)$ and fixed values K_{Dx} and β. By varying β and repeating the D-decomposition procedure, different stability regions can be obtained.

5. SIMULATION RESULTS

In this section, simulation results of system described with (15)-(16) are presented. Using the D-decomposition method, parameter space $\left(K_{P\varphi}, K_{D\varphi}\right)$ can be divided into stable and unstable regions. The stable region can be found by checking one arbitrary test point within each region, and testing the stability of polynomial (17) using the inverse Laplace transformation.

For the case $\beta \in \left[0.7,\ 1.3\right]$ and $L = 1$ stability regions are plotted as shown in Fig. 2. Stability region can then be visualized in a 3D plot as shown in Fig.3. Picking a point deep inside stability region we ensure that system is more robust with respect to parameter variations, whick implies bigger stability margins.

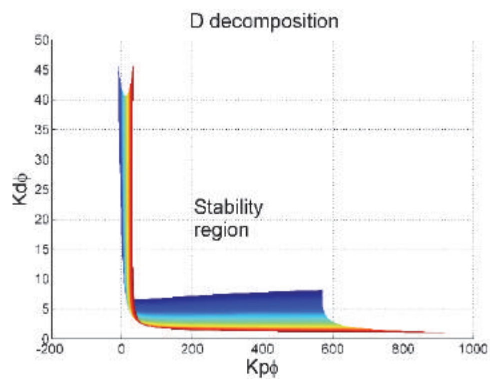

Fig. 2 *2D stability region*

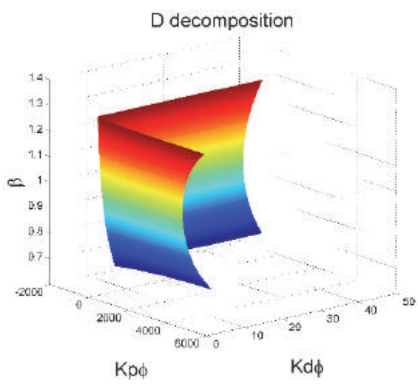

Fig. 3 *3D stability region*

Results of the swing up and stabilizing controller for $K_{Dx} = -4,\ K_{P\varphi} = 60,\ K_{D\varphi} = 6$ and $\beta = 1.1$ are shown in Fig. 4. Figure shows results for the change of pendulum angle and cart position with respect to time. Initial conditions are $\left(\dot{x}, \varphi, \dot{\varphi}\right) = \left(0,\ 162°,\ 0\right)$. A change from swing up to stabilizing controller happens when $|\varphi| < 30°$.

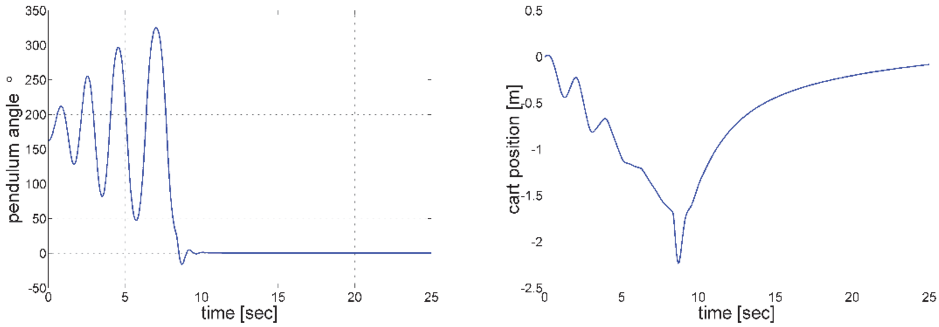

Fig. 4 *Swing up and stabilization of cart pendulum system*

6. CONCLUSION

In this paper, stability problem of cart pendulum system is investigated. Mathematical model of inverted pendulum is derived and fractional order PD controller is introduced in order to stabilize it. The problem of asymptotic stability of closed loop system is solved using the D-decomposition approach. On the basis of this method, analytical forms expressing the boundaries of stability regions in the parameters space were determined. An example is given and tests are made in order to confirm that stability domains are well calculated. Knowledge of these stability regions enables tuning of the fractional order PD controller.

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