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BRACHISTOCHRONIC MOTION OF A NONHOLONOMIC MECHANICAL SYSTEM WITH LIMITED REACTION OF CONSTRAINTS

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Abstract. The paper analyzes the problem of brachistochronic motion of a nonholonomic mechanical system, using an example of a simple car model. The system moves between two default positions at an unaltered value of the mechanical energy during motion. Differential equations of motion, containing the reaction of nonholonomic constraints and control forces, are obtained on the basis of general theorems of dynamics. Here, this is more appropriate than some other methods of analytical mechanics applied to nonholonomic systems, where the provision of a subsequent physical interpretation of the multipliers of constraints is required to solve this problem. By the appropriate choice of the parameters of state as simple a task of optimal control as possible is obtained in this case, which is solved by the application of the Pontryagin maximum principle. Numerical solution of the two-point boundary value problem is obtained by the method of shooting. Based on the thus acquired brachistochronic motion, the active control forces are determined as well as the reaction of constraints. Using the Coulomb laws of friction sliding, the minimum value of the coefficient of friction is determined to avoid car skidding at the points of contact with the ground.

1. Introduction

As is well known the classical brachistochrone problem was proposed by Johann Bernoulli in 1696 for the case of a particle moving in a vertical plane under the influence of its own gravity in a homogeneous field of gravity. Much later, the generalization of the classical brachistochrone problem was carried out within the calculus of variations [1]. A detailed review of literature related to the problems of brachistochronic motion can be found in [2] and [3]. The problems considered in the present paper involve a review of references on the Bernoulli’s case of the classical brachistochrone extended to the system of rigid bodies.

This paper, using the example of a nonholonomic mechanical system with limited reactions of constraints, presents the procedure of creating the differential equations of motion where both reactions of nonholonomic constraints and control forces figure, based on the general theorems of dynamics [5]. This paper also provides the procedure for solving brachistochronic motion of a nonholonomic mechanical system in a plane at the steady value of mechanical energy during motion, when initial and end positions are specified.

2. Description of a nonholonomic system model

In order to generate differential equations of motion of a nonholonomic mechanical system, using the example of a simplified vehicle model (see Fig. 1.). The vehicle configuration relative to the system $O\xi\eta$ is defined by a set of Lagrangian coordinates $\left(q^1,q^2,q^3,q^4\right)$, where $q^1=\xi_B$ i $q^2=\eta_B$ are Cartesian coordinates of the point *B*, $q^3=\varphi$ is the angle between the axis $O\xi$ and axis Ax, while $q^4=\theta$ is the angle between the axis Ay and the vehicle front axle axis. Further analysis refers to the case when point A cannot move in the direction of the front axle axis, while point *B* of the vehicle cannot move in the direction of the rear axle axis (lateral slipping of the front and rear axle is prevented). Such vehicle motion is limited by two ideal independent nonholonomic constraints

$$\begin{aligned} &-\dot{\xi}_B\sin\varphi+\dot{\eta}_B\cos\varphi=0,\\ &-\dot{\xi}_A\sin\left(\varphi+\theta\right)+\dot{\eta}_A\cos\left(\varphi+\theta\right)=0. \end{aligned} \tag{1}$$

During motion, the vehicle is acted on by the control force $\vec{F}_1=\vec{F}_1\left(t\right)$ along the axis Ax, as well as by the drag, proportinate to the first degree of the velocity of point C, with the coefficient of proportionality k_2, where $\vec{F}_2=-k_2\vec{V}_C$. During motion, the vehicle front axle is acted on by the control moment $L_1=L_1\left(t\right)$, around the vertical axis perpendicular to the plane of motion, the resistance moment L_2, proportionate to the relative angular velocity of axle rotation, where $L_2=k_1\dot{\theta}$, and the resistance moment L_3, proportionate to the realtive angle of the front axle rotation around the vertical axis, where $L_3=k_3\theta.$ Now, the differential equations of vehicle motion

$$\begin{aligned} &M\left[\dot{V}-\left(l_2+\frac{M_2}{M}l_1\right)\dot{\varphi}^2\right]=F_1-k_2V-R_A\sin\theta,\\ &M\left[\dot{\varphi}V+\left(l_2+\frac{M_2}{M}l_1\right)\ddot{\varphi}\right]=R_A\cos\theta+R_B-k_2l_2\dot{\varphi},\\ &J^*\ddot{\varphi}+J_2\ddot{\theta}+M\left(l_2+\frac{M_2}{M}l_1\right)\dot{\varphi}V=R_Al\cos\theta - k_2l_2^2\dot{\varphi}, \end{aligned} \tag{2}$$

where $M=M_1+M_2$, $J=J_1+J_2$ end $J^*=M_1l_2^2+M_2l^2+J$. The vectors of vehicle angular velocity and angular acceleration are $\vec{\omega}=\dot{\varphi}\vec{k}$ and $\vec{\varepsilon}=\ddot{\varphi}\vec{k}$, respectively. The differential equation of vehicle front axle rotation, has the form

$$J_2\left(\ddot{\varphi}+\ddot{\theta}\right)=L_1-k_1\dot{\theta}-k_3\theta. \tag{3}$$

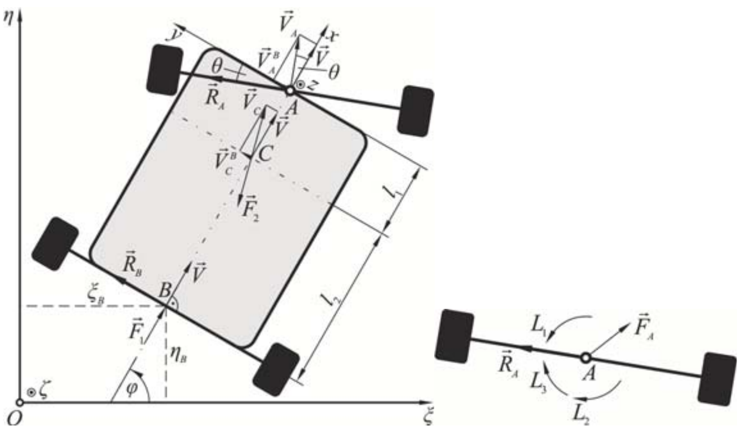
Solving the system of equations (2) and (3), the reactions of nonholonomic constraints are obtained, as well as the control force and control moment

$$\begin{aligned} &R_A=\frac{1}{l\cos\theta}\left[J^*\ddot{\varphi}+J_2\ddot{\theta}+\left(Ml_2+M_2l_1\right)\dot{\varphi}V+k_2\frac{l_2^2}{l}\tan\theta V\right],\\ &R_B=\frac{1}{l}\left[M_1l_1\dot{\varphi}V+\left(M_1l_1l_2-J\right)\ddot{\varphi}+k_2l_1l_2\dot{\varphi} - J_2\ddot{\theta}\right],\\ &F_1\left(t\right)=M\dot{V}+k_2V+\frac{\tan\theta}{l}\left[J^*\ddot{\varphi}+J_2\ddot{\theta}+k_2l_2^2\dot{\varphi}\right],\\ &L_1\left(t\right)=J_2\left(\ddot{\varphi}+\ddot{\theta}\right)+k_1\dot{\theta}+k_3\theta. \end{aligned} \tag{4}$$

Also, during vehicle brachistochronic motion the principle of the conservation of mechanical energy holds

$$\Phi\left(V,\ \dot{\varphi},\ \dot{\theta}\right) = MV^2 + J^*\dot{\varphi}^2 + 2J_2\dot{\varphi}\dot{\theta} + J_2\dot{\theta}^2 - 2T_0 = 0, \tag{5}$$

where T_0 is vehicle kinetic energy at initial time moment $t_0 = 0$.



(a) (b)

Fig. 1. (a) Simplified vehicle model; (b) front axle.

3. Brachistochronic motion as the problem of optimal control

In this section, we will formulate the problem of brachistochronic motion as the problem of optimal control. The equations of state that describe the motion of the considered system in state space can be defined in the form

$$\begin{aligned} &\dot{\xi}_B = u_1 \cos\varphi, \quad \dot{\eta}_B = u_1 \sin\varphi, \\ &\dot{\varphi} = u_2, \quad \dot{\theta} = u_3, \end{aligned} \tag{6}$$

where controls u_1, u_2 and u_3 represent the vehicle point B velocity, angular velocity, and relative angular velocity of the front axle rotation, respectively. The state coordinates ξ_B, η_B, φ and θ were determined at the initiation of motion

$$t_0 = 0,\ \xi_B\left(t_0\right) = 0,\ \eta_B\left(t_0\right) = 0,\ \varphi\left(t_0\right) = 0,\ \theta\left(t_0\right) = \theta_0, \tag{7}$$

while state coordinates ξ_B, η_B, φ and θ at the vehicle final position

$$t = t_f,\ \xi_B\left(t_f\right) = a,\ \eta_B\left(t_f\right) = b,\ \varphi\left(t_f\right) = \varphi_f,\ \theta\left(t_f\right) = \theta_f. \tag{8}$$

The brachistochrone problem of vehicle motion, described by differential equations (6), consists in determining the controls u_1, u_2 and u_3, as well as their corresponding state coordinates ξ_B, η_B, φ and θ, so that the vehicle starting from the initial state (7) moves into the final state (8), with unchanged value of mechanical energy (5), in a minimum time. This can be expressed in the form of condition, so that the functional

$$J = \int_{t_0}^{t} dt, \tag{9}$$

in the interval $\left[t_0,\, t_f\right]$ has a minimum value. In order to solve the problem of optimal control, formulated using the Pontryagin maximum principle [7], we will create Pontryagin's function in the form as follows

$$H = \lambda_0 + \lambda_x u_1 \cos\varphi + \lambda_y u_1 \sin\varphi + \lambda_\varphi u_2 + \lambda_\theta u_3 + \mu \Phi\left(u_1,\, u_2,\, u_3\right), \tag{10}$$

where $\lambda_0 = \text{const.} \le 0,\ \lambda_x,\ \lambda_y,\ \lambda_\varphi$ and λ_θ are coordinates of conjugate vector, where it can be taken that $\lambda_0 = -1$, while μ is a multiplier corresponding to (5). Based on Pontryagin's function (10), the conjugate system of differential equations is

$$\dot{\lambda}_x = 0,\ \dot{\lambda}_y = 0,\ \dot{\lambda}_\varphi = u_1\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi\right),\ \dot{\lambda}_\theta = 0. \tag{11}$$

The conditions for determining optimal control can be expressed in the form

$$\frac{\partial H}{\partial u_i} = 0,\ \frac{\partial^2 H}{\partial u_i \partial u_j} u_i u_j \le 0,\ \ \left(i,\, j{=}1{,}2{,}3\right). \tag{12}$$

Applying the Theorem 1 [7], it follows directly that the value of Pontryagin's function on the optimal trajectory equals zero, for $\forall t \in \left[t_0,\, t_f\right]$

$$-1 + \lambda_x u_1 \cos\varphi + \lambda_y u_1 \sin\varphi + \lambda_\varphi u_2 + \lambda_\theta u_3 + \mu \Phi\left(u_1,\, u_2,\, u_3\right) = 0, \tag{13}$$

Now, based on (12) and (13), we obtain the value of a multiplier μ, as well as the relations of control in the following form

$$\begin{aligned} &\mu = -\frac{1}{4T_0},\quad u_1 = \frac{2T_0}{M}\left(\lambda_x \cos\varphi + \lambda_y \sin\varphi\right),\\ &u_2 = \frac{2T_0}{J^* - J_2}\left(\lambda_\varphi - \lambda_\theta\right),\quad u_3 = \frac{2T_0}{J^* - J_2}\left(\frac{J^*}{J_2}\lambda_\theta - \lambda_\varphi\right). \end{aligned} \tag{14}$$

Based on relation (13) determined at the initial time moment, and based on (7), (12) and (14), we obtain the value of coordinate λ_φ at the initial time moment, as well as the value of the constant λ_x

$$\lambda_\varphi\left(t_0\right) = \lambda_\theta \pm \sqrt{\frac{\left(J^* - J_2\right)}{4T_0^2}\left(2T_0 - MV_0^2 - \frac{4T_0^2}{J_2}\lambda_\theta^2\right)},\quad \lambda_x = \frac{MV_0}{2T_0}, \tag{15}$$

where V_0 is the velocity of point B at the initial time moment. The shooting method was used in numerical procedure for solving the corresponding two-point boundary value problem, based on (6), (7), (8), (11), (14) and (15), [8]. Shooting consists in determining the unknown coordinates of conjugate vector λ_x, λ_y and λ_θ, having in mind (15), as well as a minimum required time t_f, so that the vehicle starting from initial state (7) moves into the final state (8). The two-point boundary value problem was solved for the following values of parameters

$$\begin{aligned} &T_0 = 1000\frac{\text{kgm}^2}{\text{s}^2},\ \theta_0 = \frac{\pi}{30}\text{rad},\ \theta\left(t_f\right) = 1\text{rad},\ \varphi\left(t_f\right) = \frac{\pi}{2}\text{rad},\ M_1 = 1000\text{kg},\\ &M_2 = 110\text{kg},\ J_1 = 1500\text{kgm}^2,\ J_2 = 30\text{kgm}^2,\ l_1 = 0.75\text{m},\ l_2 = 1.65\text{m},\ a = 5\text{m},\\ &b = 5\text{m},\ k_1 = 0.5\text{Nsm},\ k_2 = 100\frac{\text{Ns}}{\text{m}},\ k_3 = 100\text{Nm}. \end{aligned} \tag{16}$$

In accordance with (9), the time of the vehicle brachistochronic motion, as well as the conjugate vector coordinates, for the given values of the system parameters (16), are $t_f = 6.2296\text{s}$, $\lambda_x = 0.5117$, $\lambda_y = 0.5117$ i $\lambda_\theta = 0.0059$.

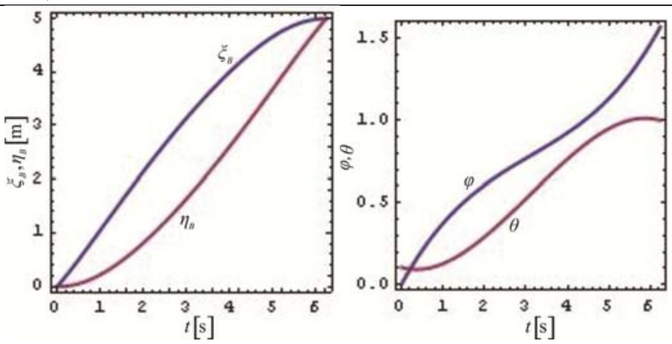


Fig. 2. State coordinates $\xi_B(t)$, $\eta_B(t)$, $\varphi(t)$ and $\theta(t)$.

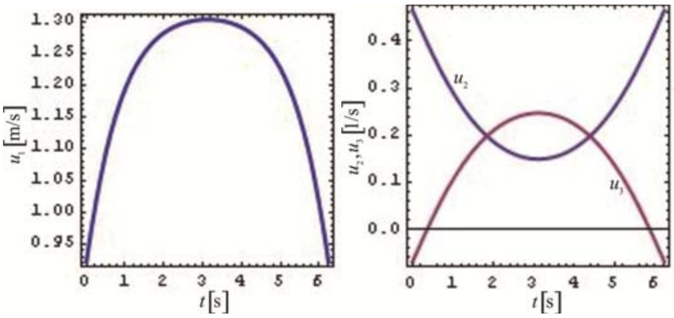


Fig. 3. Optimal controls $u_1(t)$, $u_2(t)$ and $u_3(t)$.

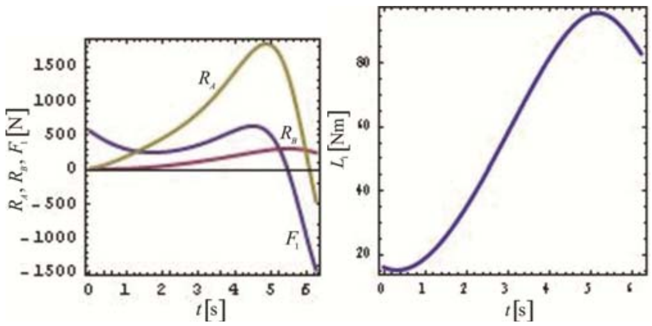


Fig. 4. Reactions of constraints $R_A(t)$, $R_B(t)$, and control forces $F_1(t)$, $L_1(t)$.

4. Conditions for constraints based on Coulomb sliding friction

The necessary dynamic conditions for realizing such motion [6], and based on the Coulomb laws of sliding friction, are

$$\sqrt{R_B^2 + F_1^2} \leq N_1\mu_1, \quad |R_A| \leq N_2\mu_2, \tag{17}$$

where μ_1 and μ_2 are the coefficients of sliding friction between rear and front wheels, respectively, and stationary surface. Normal reactions of rear and front axle stationary surface are $N_1 = 3065.6\text{N}$ and $N_2 = 7823.5\text{N}$, respectively. The diagrams below, based on above considerations, show the laws of minimum required values of the coefficients of sliding friction μ_1^* and μ_2^* in function of time.

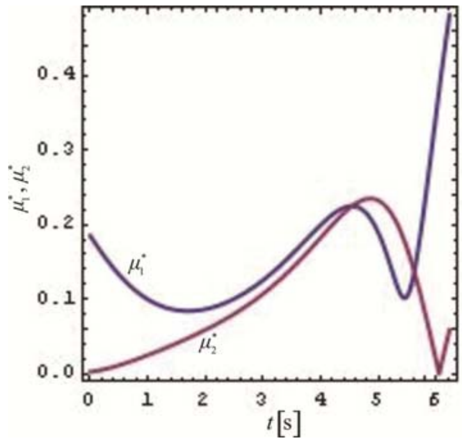


Fig. 5. Diagrams $\mu_1^* = \mu_1^*(t)$ and $\mu_2^* = \mu_2^*(t)$.

Based on above considerations, it can be inferred (see Fig. 5.) that a minimum required value of the coefficient of sliding friction, between stationary surface and vehicle wheels, is $\mu^* \geq 0.5$.

5. Conclusions

Applying the Coulomb laws of sliding friction, minimum required values for the coefficient of sliding friction were determined, so as to prevent slipping of both vehicle rear and front axle, as presented in Fig. 5. Authors consider that results obtained in this paper can be extended to the case when the coefficients of sliding friction are below minimum required values. In that case, as well as in the case when control forces are constrained, the problem of optimal control becomes considerably more complex, which will be the subject of future studies.

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