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REALIZATION OF THE BRACHISTOCHRONIC MOTION OF A NONHOLONOMIC VARIABLE MASS MECHANICAL SYSTEM BY IDEAL HOLONOMIC CONSTRAINT

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Abstract:

The paper considers realization of the brachistochronic motion of a nonholonomic mechanical system, composed of variable mass particles, by means of an ideal holonomic constraint. It is assumed that the system performs planar motion in an arbitrary field of forces and that it has two degrees of freedom. In addition, the laws of the time-rate of mass variation of the particles, as well as relative velocities of the expelled and gained particles, respectively, are known. The first time-derivative of quasi-velocity is taken as control variable. Applying Pontryagin's maximum principle and singular optimal control theory, the problem of brachistochronic motion is solved as a two-point boundary value problem (TPBVP). The considerations are illustrated via an example.

Key words: brachistochrone, variable mass, mechanical system, nonholonomic constraints, holonomic constraints, Pontryagin's maximum principle, optimal control

1. Formulation of the problem

Consider planar motion of the mechanical system composed of N material points. Without loss of generality all material points can be of variable mass. The system configuration is defined by means of n generalized coordinates $\boldsymbol{q}=\left(q^1, q^2,..., q^n\right)^T$, which are geometrically independent, and based on them the mechanical system position is unambiguously determined. In addition, the laws of the time-rate of mass variation of the material points can be considered to be known:

$$m_l = m_l(t), \qquad l = 1,..., N, \tag{1}$$

where $m_l(t)$ are continuous and differentiable functions of time. Mass variation can be realized by expelling or gaining of masses, assuming that the process of expelling and gaining of masses, respectively, is continuous over the considered interval of time.

Relative velocities of expelling and gaining of masses, respectively, are considered to be known:

$$\vec{v}_l^{rel} = \vec{v}_l^{rel}(\boldsymbol{q},\dot{\boldsymbol{q}},t), \qquad l = 1,\ldots, N, \tag{2}$$

where $\dot{\boldsymbol{q}}=\left(\dot{q}^1, \dot{q}^2,..., \dot{q}^n\right)^T$ is the vector of generalized velocities. Also, the well known Einsten summation convention is deployed in the paper, where the indices have a range of values as follows: $i, j, k, r = 1,\ldots,n;$ $\alpha,\beta,\gamma,\delta = 1, 2;$ $v,\rho = 3,\ldots,n$. Planar motion of the considered mechanical system is constrained by p ideal independent stationary nonholonomic homogeneous constraints of the form:

$$\gamma^{v}\left(\boldsymbol{q},\dot{\boldsymbol{q}}\right) \equiv \dot{q}^{v} - c_{\alpha}^{v}\dot{q}^{\alpha} = 0, \tag{3}$$

where $c_{\alpha}^{v} = c_{\alpha}^{v}\left(\boldsymbol{q}\right)$. Number p is taken in such way that the number of degrees of freedom of a mechanical system motion is m=n-p=2, and therefore p=n-2. At the same time, m=2 represents the number of kinematically independent coordinates q^{α}, which correspond to independent generalized velocities $\dot{q}^{\alpha}$ that can be expressed as a linear form of independent quasi-velocities V^{β} [1, 2]:

$$\dot{q}^{\alpha} = b_{\beta}^{\alpha}V^{\beta}. \tag{4}$$

If (3) and (4) are taken into account, dependent generalized velocities can be written as follows:

$$\dot{q}^{v} = b_{\beta}^{v}V^{\beta}, \tag{5}$$

where $b_{\beta}^{v} = c_{\alpha}^{v}b_{\beta}^{\alpha}$. The kinetic energy of a nonholonomic scleronomic mechanical system is a homogeneous quadratic form of independent quasi-velocities: [1, 2, 3, 4]:

$$T^{*} = \frac{1}{2}G_{\alpha\beta}V^{\alpha}V^{\beta}, \tag{6}$$

where:

$$G_{\alpha\beta}\left(\boldsymbol{q},t\right) = a_{ij}b_{\alpha}^{i}b_{\beta}^{j} \tag{7}$$

and where $G_{\alpha\beta}$ are the covariant coordinates of metric tensor relative to kinematically independent coordinates q^{α} taking into account independent quasi-velocities V^{α}, $a_{ij} = a_{ij}\left(\boldsymbol{q}, t\right)$ are the covariant coordinates of metric tensor of the function of generalized coordinates and time t, $b_{\alpha}^{i} = b_{\alpha}^{i}\left(\boldsymbol{q}\right)$ are continuous functions with continuous first derivatives in the area of mechanical system considerations. It can be considered that the studied mechanical system is moving in a field of known potential forces, whose potential energy equals:

$$\Pi = \Pi(\boldsymbol{q}, t), \tag{8}$$

and that the system is acted on by known arbitrary nonpotential forces, so that the generalized forces are:

$$Q_{i}^{w} = Q_{i}^{w}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{9}$$

The differential equations of motion for the considered system, as a function of kinematically independent coordinates, are written in covariant form [1, 2, 3, 4]:

$$G_{\alpha\beta}\dot{V}^{\beta} = \Delta_{\alpha}, \tag{10}$$

where:

$$\Delta_{\alpha}(\boldsymbol{q}, \boldsymbol{V}, t) = \tilde{Q}_{\alpha} - a_{ij}b_{\alpha}^{i}b_{\gamma}^{k}\left(\frac{\partial b_{\beta}^{j}}{\partial q^{k}} + \Gamma_{kr}^{j}b_{\beta}^{r}\right)V^{\gamma}V^{\beta}, \tag{11}$$

whereas the generalized forces corresponding to kinematically independent coordinates are represented as:

$$\tilde{Q}_{\alpha}\left(\boldsymbol{q}, \boldsymbol{V}, t\right) = b_{\alpha}^{i}Q_{i}, \tag{12}$$

where $\boldsymbol{V} = \left(V^{1}, V^{2}\right)^{T}$, Q_{i} are covariants generalized forces corresponding to geometrically independent coordinates, Γ_{kr}^{j} are Christoffel symbols of the second kind. The generalized forces corresponding to geometrically independent coordinates can be represented, in a general case, in the form as follows [5, 6]:

$$Q_i(\boldsymbol{q},\dot{\boldsymbol{q}},t) = -\frac{\partial \Pi}{\partial q^i} + Q_i{}^w + Q_i{}^{\text{var}} + Q_i{}^c + Q_i{}^{\Lambda}. \tag{13}$$

The generalized reaction forces that develop due to expelling and gaining of masses, respectively, can be written as [5, 6]:

$$Q_i^{\text{var}}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \sum_{l=1}^{N} \dot{m}_l \vec{v}_l^{rel} \cdot \frac{\partial \vec{r}_l}{\partial q^i}, \tag{14}$$

while at the same time $Q_i^c = Q_i^c(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ are generalized control forces, whose total power during brachistochronic motion equals zero:

$$Q_i^c \dot{q}^i = 0, \tag{15}$$

where, in accordance with (7) and (17) it can be written:

$$Q_\alpha^c V^\alpha = 0. \tag{16}$$

Since generalized forces due to imposed nonholonomic constraints (4) can be written in the form as follows:

$$Q_i^\Lambda(\boldsymbol{q},\dot{\boldsymbol{q}}) = \Lambda_\nu \frac{\partial \gamma^\nu}{\partial \dot{q}^i}, \tag{17}$$

where Λ_ν are Lagrange’s multipliers of the constraints, based on (3), (12) and (17), it can be shown that:

$$\tilde{Q}_\alpha^\Lambda = b_\alpha^\beta Q_\beta^\Lambda + b_\alpha^\nu Q_\nu^\Lambda = \Lambda_\nu \left(b_\alpha^\nu - b_\alpha^\beta c_\beta^\nu \right) = 0. \tag{18}$$

Based on these equations, it can be concluded that Lagrange’s multipliers of the constraints do not occur in differential equations of motion (15), and hence the procedure of defining the reactions of nonholonomic constraints is completely separated from the procedure of defining the system motion. Multiplying both sides of the equation (10) by V^α and summing over an index α it is obtained:

$$G_{\alpha 1}\dot{V}^1 V^\alpha + G_{\alpha 2}\dot{V}^2 V^\alpha = \Delta_\alpha V^\alpha. \tag{19}$$

The last relation enables to express one arbitrary second time-derivative of a quasi-coordinate as a function of the other one. Taking into account (16), in equations (19) the generalized control forces will not figure. Now, relation (19) can be expresed in the following form [3, 4]:

$$\dot{V}^2 = \Theta + \Theta_1 \dot{V}^1, \tag{20}$$

where:

$$\Theta(\boldsymbol{q},\boldsymbol{V},t) = \frac{\Delta_\beta V^\beta}{G_{\alpha 2} V^\alpha}, \quad \Theta_1(\boldsymbol{q},\boldsymbol{V},t) = -\frac{G_{\beta 1} V^\beta}{G_{\alpha 2} V^\alpha}, \tag{21}$$

where it is assumed that the expression $G_{\alpha 2} V^\alpha$ which figures in the denominators of the relations (21) does not equal zero during the system motion.

The question is posed on realizing the motion of the presented mechanical system. The answer is found in subsequently imposed ideal holonomic constraint. Since it is the mechanical system with kinematically independent generalized coordinates, the motion can be realized by the imposition of smooth guides to a single particle, whose motion is defined by previous numerical integration of differential equations. Without loss of generality, let it be point C of the system. This way, the brachistochronic motion is realized without active forces' influence, which is in accordance with the elementary brachistochrone problem of aparticle in a vertical plane.

Let the values of generalized coordinates be specified, as well as the value of mechanical energy of the mechanical system at the initial instant of time:

$$t_0 = 0, \quad \boldsymbol{q}(t_0) = \boldsymbol{q}_0, \tag{22}$$

$$T^*\left(\boldsymbol{q}_0, \boldsymbol{V}_0, t_0\right) + \Pi\left(\boldsymbol{q}_0, t_0\right) = E_0, \tag{23}$$

and also the values of generalized coordinates corresponding to the final position of the system:

$$\boldsymbol{q}\left(t_f\right) = \boldsymbol{q}_f, \tag{24}$$

where $E_0 \in \mathbb{R}$ and $t_{\mathrm{f}} \in \mathbb{R}$. The problem of brachistochronic planar motion of a variable mass nonholonomic mechanical system, whose differential equations of motion are given in the form (10), consists of defining the generalized control forces $Q_i^c = Q_i^c(t)$, and corresponding equations of the system motion $q_i = q_i(t)$, so that the system moves in the minimum time t_f from the initial state defined by (22) and (23) to the final position defined by (24).

2. Brachistochrone problem as an optimal control task

The presented brachistochrone problem can be formulated as a task of optimal control by introducing scalar control u [3, 4]:

$$u = \dot{V}^1, \tag{25}$$

The normal form of first-order differential equations, known in the optimal control theory as the state equations, can be written by incorporating the rheonomic coordinate $q^{n+1} \triangleq t$ in the following manner:

$$\begin{aligned}
\dot{q}^i &= f_i(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, u) \equiv b_\alpha^i V^\alpha, \\
\dot{q}^{n+1} &= f_{n+1}(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, u) \equiv 1, \\
\dot{V}^1 &= f_{(1)}(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, u) \equiv u, \\
\dot{V}^2 &= f_{(2)}(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, u) \equiv \Theta(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}) + \Theta_1(\boldsymbol{q}, \boldsymbol{V}, q^{n+1})u.
\end{aligned} \tag{26}$$

The brachistochrone problem of the considered nonholonomic system motion described by the state equations (26), consists of defining the optimal scalar control u and corresponding optimal trajectories in state space $q^i(t)$, so that the mechanical system moves from the initial state defined by (22) and (23) to the final position (24), in the minimum time, which can be expressed using conditions for the functional [7]:

$$J\left(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, u\right) = \int_0^{t_f} dt, \tag{27}$$

over the interval $[0, t_f]$ it has minimum value. In order to solve the problem of optimal control by applying Pontryagin's maximum principle [8], the Hamiltonian is created of the Hamilton-Pontryagin form:

$$H\left(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, u, \boldsymbol{\lambda}, \boldsymbol{\nu}\right) = -1 + \lambda_j b_\alpha^j V^\alpha + \lambda_{n+1} + \nu_1 u + \nu_2(\Theta + \Theta_1 u), \tag{28}$$

where $\boldsymbol{\lambda} = \left(\lambda_1, \lambda_2, ..., \lambda_{n+1}\right)^T$, $\boldsymbol{\nu} = \left(\nu_1, \nu_2\right)^T$, whereas $\lambda_i(\cdot):\left[0, t_f\right] \to \mathbb{R}$, $\lambda_{n+1}(\cdot):\left[0, t_f\right] \to \mathbb{R}$ and $\nu_\alpha(\cdot):\left[0, t_f\right] \to \mathbb{R}$ are costate variables, so that the costate system of differential equations has the form:

$$\begin{aligned}
\dot{\lambda}_i &= -\frac{\partial H}{\partial q^i} = -\lambda_j \frac{\partial b_\alpha^j}{\partial q^i} V^\alpha - \nu_2\left(\frac{\partial \Theta}{\partial q^i} + \frac{\partial \Theta_1}{\partial q^i} u\right), \\
\dot{\lambda}_{n+1} &= -\frac{\partial H}{\partial q^{n+1}} = -\lambda_j \frac{\partial b_\alpha^j}{\partial q^{n+1}} V^\alpha - \nu_2\left(\frac{\partial \Theta}{\partial q^{n+1}} + \frac{\partial \Theta_1}{\partial q^{n+1}} u\right), \\
\dot{\nu}_\beta &= -\frac{\partial H}{\partial V^\beta} = -\lambda_j b_\beta^j - \nu_2\left(\frac{\partial \Theta}{\partial V^\beta} + \frac{\partial \Theta_1}{\partial V^\beta} u\right).
\end{aligned} \tag{29}$$

Based on (28), it can be written:

$$H\left(\boldsymbol{q},\boldsymbol{V},q^{n+1},u,\boldsymbol{\lambda},\boldsymbol{\nu}\right)=H_0+H_1u, \tag{30}$$

where:

$$H_0=-1+\lambda_j c_\alpha^j V^\alpha+\lambda_{n+1}+\nu_2\Theta,$$
$$H_1=\nu_1+\nu_2\Theta_1. \tag{31}$$

For the case of control known in the optimal control theory as a singular control, where control figures linearly in the state equations, the necessary optimality condition of Pontryagin's maximum principle is of the form as follows [9]:

$$\frac{\partial H}{\partial u}=H_1=0, \tag{32}$$

from where singular optimal control u cannot be explicitly defined. Hence, it is required that H_1 be identically equal to zero alongside the optimal trajectory of state. Singular optimal control u is defined by further differentiation with respect to time (32) taking into account (26) and (29):

$$\frac{d^k}{dt^k}\left[\frac{\partial H}{\partial u}\right]=0,\quad k=0,1,2,... \tag{33}$$

In defining the relations (33) the Poisson bracket formalism will be applied [10]:

$$\dot{H}_1=\left\{H,H_1\right\}=\left\{H_1,H_0\right\}+\left\{H_1,H_1\right\}u=0. \tag{34}$$

Taking into account (42), as well as that $\left\{H_1,H_1\right\}=0$ [10], it is obtained:

$$\left\{H_1,H_0\right\}=\sum_{\theta=1}^{n+3}\left(\frac{\partial H_1}{\partial y^\theta}\frac{\partial H_0}{\partial \zeta^\theta}-\frac{\partial H_1}{\partial \zeta^\theta}\frac{\partial H_0}{\partial y^\theta}\right)=0, \tag{35}$$

where $\boldsymbol{y}=(y^1,y^2,\ldots,y^{n+3})^T\triangleq(q^1,q^2,\ldots,q^{n+1},V^1,V^2)^T$ and $\boldsymbol{\zeta}=(\zeta^1,\zeta^2,\ldots,\zeta^{n+3})^T\triangleq(\lambda_1,\lambda_2,\ldots,\lambda_{n+1},\nu_1,\nu_2,)^T$. Further differentiation (35) yields:

$$\left\{\left\{H_1,H_0\right\},H_0\right\}+\left\{\left\{H_1,H_0\right\},H_1\right\}u=0. \tag{36}$$

From where singular control can be expressed as:

$$u=-\frac{\left\{\left\{H_1,H_0\right\},H_0\right\}}{\left\{\left\{H_1,H_0\right\},H_1\right\}}. \tag{37}$$

Furthermore, the transversality conditions can be represented in the form as follows:

$$\left(\lambda_i\Delta q^i+\lambda_{n+1}\Delta q^{n+1}+\nu_\alpha\Delta V^\alpha\right)\Big|_0^{t_f}=0, \tag{38}$$

$$\left(H\Delta t\right)\Big|_0^{t_f}=0, \tag{39}$$

where $\Delta(\cdot)$ is asynchronous variation [1, 2] of the quantity $(\cdot)$. Based on condition (32), the costate variable ν_1 can be expressed as a function of the costate variable ν_2:

$$\nu_1=-\nu_2\Theta_1. \tag{40}$$

Now, from equations (35), taking into account (31) and (40),one can express:

$$\lambda_1=\lambda_1(\boldsymbol{q},\boldsymbol{V},q^{n+1},\lambda_2,\lambda_3,...,\lambda_n,\nu_2). \tag{41}$$

Since the initial position of the mechanical system according to (22) is defined, it follows:

$$\Delta t(t_0)=0,\qquad \Delta q^i(t_0)=0,\qquad \Delta q^{n+1}(t_0)=0. \tag{42}$$

If (42) is taken into account and the operator of asynchronous variation is applied to (23), it can be obtained:

$$G_{\alpha\beta}(t_0)V^\beta(t_0)\Delta V^\alpha(t_0)=0, \tag{43}$$

and lastly, after substituting (40) and (42) into (38),it is obtained:

$$\nu_\alpha(t_0)\Delta V^\alpha(t_0)=\nu_2(t_0)G_{\alpha\beta}(t_0)V^\beta(t_0)\Delta V^\alpha(t_0)=0. \tag{44}$$

Based on (42), (43) and (44), it is obvious that the transversality conditions (38) and (39) in the initial configuration of the system are satisfied. In the final configuration (24) of the mechanical system the time is not known, and based on it, the transversality condition results from (39):

$$H\left(t_f\right) = 0, \tag{45}$$

and as quantities $V^{\alpha}(t_f)$ and $\Delta q^{n+1}(t_f)$ are not a priori defined ($\Delta V^{\alpha}(t_f) \neq 0, \Delta q^{n+1}(t_f) \neq 0$), the next transversality conditions are obtained from (38):

$$\nu_{\alpha}(t_f) = 0,\ \lambda_{n+1}(t_f) = 0. \tag{46}$$

Based on (28), (41), (45) and (46), the following dependence can be established in analytical form:

$$\lambda_e(t_f) = \lambda_e(\boldsymbol{V}(t_f), q^{n+1}(t_f), \lambda_{i,i\neq 1,i\neq e}(t_f)), \tag{47}$$

where $\boldsymbol{V}(t_f) = (V^1(t_f), V^2(t_f))^T$ and $q^{n+1}(t_f) = t_f$.

If considerations are restricted to the first order singular controls, where $\{\{H_1, H_0\}, H_1\} \neq 0$, using (40) and (41), singular scalar control u_{sing} from (37) can be represented in the form as follows:

$$u_{sing} = u_{sing}(\boldsymbol{q}, \boldsymbol{V}, q^{n+1}, \lambda_2, \lambda_3, ..., \lambda_n, \nu_2). \tag{48}$$

Also, the Kelley necessary condition for the first order singular control is given in the form [9]:

$$-\frac{\partial}{\partial u}\left(\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial u}\right]\right) \leq 0 \tag{49}$$

Applying the Poisson brackets, this condition is reduced to:

$$K = \{\{H_1, H_0\}, H_1\} > 0 \tag{50}$$

Substituting (48) into (26) and (29) yields a two-point boundary value problem (TPBVP) with 2*n*+3 first-order nonlinear normal form differential equations. Due to nonlinearity, in a general case, it is necessary to apply the appropriate numerical method [11]. In this paper, the shooting method will be deployed. The shooting method is most suitable to perform in this case by the backward numerical integration choosing the (*n*+1) values $\lambda_{i,i\neq 1,i\neq e}(t_f), V^{\alpha}(t_f), t_f$, which will ensure fulfillment of the same number of initial conditions (22) and (23). The value $\lambda_1(t_f)$ was defined via (41) for $t=t_f$, and $\lambda_e(t_f)$ from the expression (47).

Numerical solution of the problem can be performed applying a software package *Wolfram Mathematica* [12] in two steps. In step 1 numerical relations are established in the form of the system of differential equations with unknown values that are chosen. In establishing these relations the functions *NDSolve*[] and *First*[] are employed. In step 2 unknown boundary values are defined by applying the function *FindRoot*[]. After the appropriate boundary values are defined, the system of differential equations is solved by applying the function *NDSolve*[]. Thus, the given problem is solved and will be presented using an example.

4. Numerical example

The example shows a nonholonomic mechanical system composed of two variable mass material points *A* and *B* with an imposed constraint of motion in the form of perpendicularity of the velocities by means of Chaplygin blades of negligible masses, as indicated by Fig. 1. In step 1, for the needs of further considerations, two Cartesian coordinate reference systems must be introduced. The first, a stationary coordinate system *Oxyz*, whose coordinate plane *Oxy* coincides with the horizontal plane of motion, and the second, a non-stationary coordinate system *Bξηζ*, whose coordinate origin is attached to point *B* of the system, the coordinate plane *Bξη* coinciding with the plane *Oxy*. In addidtion, the axis of the non-stationary coordinate system *Bξ* is defined by the direction BA, that is, $A \in B\xi$. Unit vectors of the non-stationary coordinate system axes are $\vec{\lambda}, \vec{\mu}$ and $\vec{v}$, respectively. Variable-mass material points A and B are interconnected by a lightweight mechanism of the 'pitchfork' type, which allows the distance between the points to change, i.e. $\overline{BA} = \xi \neq \text{const}$.

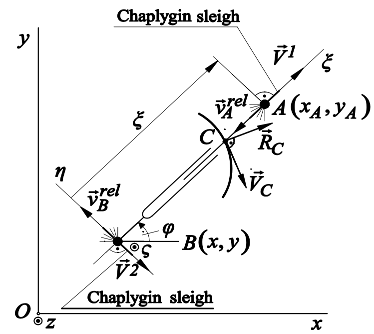


Figure 1. Variable mass nonholonomic mechanical system.

The configuration of the considered system is defined by a set of Lagrangian coordinates $\boldsymbol{q} = \left(q^1, q^2, q^3, q^4\right)^T$, where $q^1 \triangleq x$ and $q^2 \triangleq y$ are Cartesian coordinates of the point B, $q^3 \triangleq \varphi$ is the angle between the axes Ox and $B\xi$ and $q^4 \triangleq \xi$ is the relative coordinate of the point A relative to the non-stationary coordinate system.

Changes in masses of the points A and B are specified in the following form:

$$m_A(t) = m_0 e^{-k_A t}, \quad m_B(t) = m_0 e^{-k_B t}, \tag{51}$$

where m_0 is mass of the points A and B at the initial instant of time, and k_A and k_B are defined positive constants. Without loss of generality, the magnitudes of relative velocities of the particles' expelling from the points A and B are constant and mutually equal:

$$v_A^{rel} = v_B^{rel} = v_r, \tag{52}$$

where v_r is a defined positive constant, and $\vec{v}_A^{rel} = -v_r \vec{\lambda}$ and $\vec{v}_B^{rel} = v_r \vec{\mu}$. According to the restriction of motion of the points A and B, and in accordance with (3), nonholonomic homogeneous constraints can be written in the following manner:

$$\begin{aligned} \gamma^3 &\equiv \dot{q}^1 \cos(q^3) + \dot{q}^2 \sin(q^3) = 0, \\ \gamma^4 &\equiv -\dot{q}^1 \sin(q^3) + \dot{q}^2 \cos(q^3) + q^4 \dot{q}^3 = 0. \end{aligned} \tag{53}$$

For independent quasi-velocities, the velocities of the points A and B are taken:

$$\begin{aligned} V^1 &= V_A = \dot{q}^4, \\ V^2 &= V_B = \dot{q}^1 \sin(q^3) - \dot{q}^2 \cos(q^3). \end{aligned} \tag{54}$$

Now, according to (4), (5), (53) and (54), all generalized velocities can be expressed via independent quasi-velocities:

$$\begin{aligned} \dot{q}^1 &= \sin(q^3) V^2, \\ \dot{q}^2 &= -\cos(q^3) V^2, \\ \dot{q}^3 &= \frac{1}{q^4} V^2, \\ \dot{q}^4 &= V^1. \end{aligned} \tag{55}$$

The kinetic energy of the system, according to (6), is written in the following form:

$$T^{*}=\frac{1}{2}\left(m_{A}V_{A}^{2}+m_{B}V_{B}^{2}\right). \tag{56}$$

At point C of the system, an ideal holonomic stationary constraint is imposed in the form of smooth guides, so control was accomplished without active control forces by means of the constraint reaction $\vec{R}_C$, whose components are $\vec{F}_1=F_1(t)\vec{\lambda}$ and $\vec{F}_2=F_2(t)\vec{\mu}$ realizing the constraint in such way that the condition $\vec{R}_C\cdot\vec{v}_C=0$, i.e. $F_1V^1-F_2\frac{AC}{q^4}V^2=0$ is satisfied during brachistochronic motion. Accordingly, the line of the guide path coincides with with the line of the point C path, positioned in the AB direction, and therefore the parametric equations of the guide line are specified in the form as follows:

$$\begin{aligned}x_C(t)&=q^1+\left(q^4-\overline{AC}\right)\cos(q^3),\\ y_C(t)&=q^2+\left(q^4-\overline{AC}\right)\sin(q^3).\end{aligned} \tag{57}$$

Now, based on (10), (13), (14), (18), (54) and (55), differential equations of motion of the system can be constructed:

$$\begin{aligned}m_A\dot{V}^1&=k_Am_Av_r+F_1,\\ m_B\dot{V}^2&=k_Bm_Bv_r-F_2\frac{AC}{q^4}.\end{aligned} \tag{58}$$

Also, based on (11), (21) and (58), relations Θ and Θ_1 are of the form:

$$\begin{aligned}\Theta_1&=-\frac{m_A}{m_B}\frac{V^1}{V^2},\\ \Theta&=k_A\frac{m_A}{m_B}\frac{V^1}{V^2}v_r+k_Bv_r,\end{aligned} \tag{59}$$

therefore, according to (20), $\dot{V}^2$ can be expressed in the following form:

$$\dot{V}^2=k_A\frac{m_A}{m_B}\frac{V^1}{V^2}v_r+k_Bv_r-\frac{m_A}{m_B}\frac{V^1}{V^2}\dot{V}^1. \tag{60}$$

Afterwards, a rheonomic coordinate can be introduced and (28) and (29) can be defined applying (55) and (60), as well as all other needed quantities so as to solve the formulated problem.

For initial and end conditions (22), (23) and (24) it is taken:

$$\begin{aligned}&t_0=0,\quad q^1(t_0)=0,\ q^2(t_0)=0,\ q^3(t_0)=0,\ q^4(t_0)=a,\\ &T^{*}(t_0)+\Pi(t_0)=\frac{1}{2}\left(m_A(t_0)V_A^2(t_0)+m_B(t_0)V_B^2(t_0)\right)=E_0,\\ &q^1(t_f)=2a,\ q^2(t_f)=-1.5a,\ q^3(t_f)=\pi/2,\ q^4(t_f)=3a.\end{aligned} \tag{61}$$

Using the numerical procedure described in the preceding Section, the solution of the problem was found for the following parameters:

$$E_0=80\frac{\text{kgm}^2}{\text{s}^2},\ a=1\text{m},\ k_A=0.35\frac{1}{\text{s}},\ k_B=0.25\frac{1}{\text{s}},\ v_r=1\frac{\text{m}}{\text{s}},\ m_0=50\text{kg},\ \overline{AC}=1/3m. \tag{62}$$

The numerical procedure gives solutions for the system of differential equations of motion, as well as for the costate system in numerical form:

$$q^1(t),\ q^2(t),\ q^3(t),\ q^4(t),\ V^1(t),\ V^2(t),\ \lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \nu_1(t),\ \nu_2(t), \tag{63}$$

and the time of brachistochronic motion t_f. Figure 2 shows trajectories of the points A, B and C.

Figure 3 shows graphic representation of the values of velocities V^1 and V^2. Figure 4 displays graphs of control forces F_1 and F_2. The control forces F_1 and F_2 can be expressed in the following form:

$$F_1 = m_A\left(u - k_A v_r\right), \quad F_2 = m_A \frac{q^4 V^1}{\overline{AC} V^2}\left(u - k_A v_r\right). \tag{64}$$

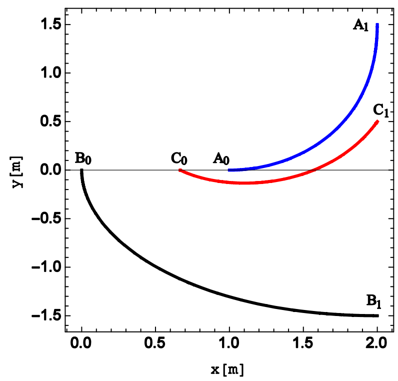


Figure 2. Graphs of the trajectories of the points A, B and C.

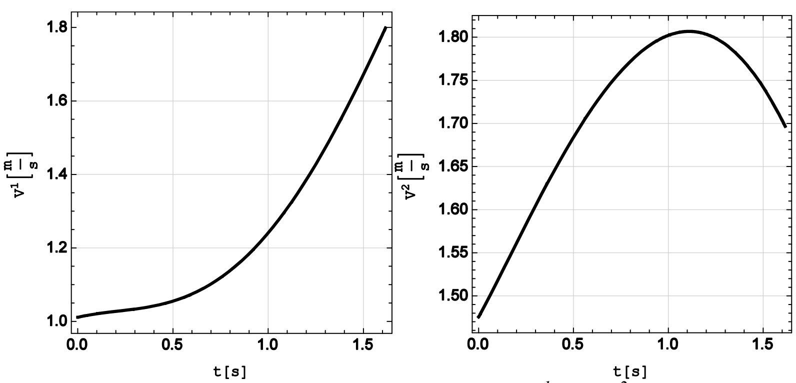


Figure 3. Graphs of the velocities V^1 and V^2.

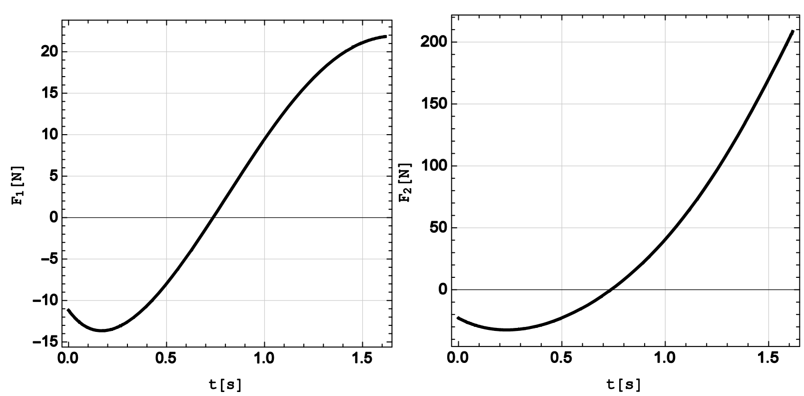
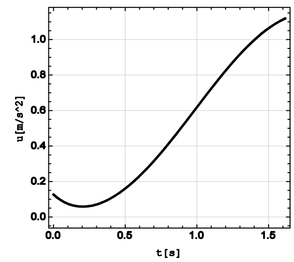


Figure 4. Graphs of control forces F_1 and F_2.

Figure 5 gives a graphic representation of the control $u(t)$. Since control in this example is the first order singular control, it is needed to satisfy Kelley's optimality condition (50). Figure 6 presents the law of change in the function K, which indicates the fulfillment of Kelley's optimality condition.



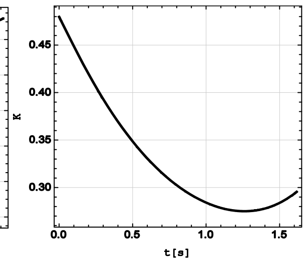


Figure 5. Graph of control $u(t)$. Figure 6. Evidence of Kelley's optimality condition.

3. Conclusions

The present work has solved the problem of realizing brachistochronic planar motion of a nonholonomic variable mass mechanical system by means of an ideal holonomic constraint. Considerations presented in this work rely on the papers [3] and [13] and thus are a kind of continuation of mentioned studies. The considered system has two degrees of freedom so that the motion can be realized by means of a single ideal holonomic constraint. The first time-derivative of quasi-velocity is taken as control variable. The brachistochrone problem is formulated as an optimal control task. Further research will go towards limiting constraint reaction, whereby non-singular controls occur.

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References

[1] Lurie, A. I.: *Analytical mechanics*, Berlin Heidelberg: Springer-Verlag, 2002.
[2] Papastavridis, J. G.: *Analytical mechanics*, New York: Oxford University Press, 2002.
[3] Jeremić B., Radulović R., Obradović A., Šalinić S., Dražić M.: *Brachistochronic motion of a nonholonomic variable-mass mechanical system in general force fields*, - Mathematics and Mechanics of Solids, Vol 24, 2019, pp. 281–298.
[4] Radulović, R.: *Global minimum time for the motion of mechanical systems with limited controls and constraint*, - Doctoral dissertation, Faculty of Mechanical engineering, University of Belgrade, 2017.
[5] Pesce, C. P.: *The application of Lagrange equations to mechanical systems with mass explicitly dependent on position*, - JAppl Mech (Trans ASME), Vol 70,2003, pp. 751–756.
[6] Cvetićanin, L.: *Conservation laws in systems with variable mass*, - *J Appl Mech* (*Trans ASME*), Vol 60, 1993, pp. 954–958.
[7] Kirk, D. E.: *Optimal control theory: An introduction*, -New York: Dover Publications, 2004.
[8] Pontryagin, LS, Boltyanskii, VG, Gamkrelidze, RV, et al.: *The mathematical theory of optimal processes*, New York: Interscience, 1962.
[9] Gabasov, R., Kirillova, F.M.: *Singular optimal controls*, Nauka, Moscow, 1973.
[10] Gabasov, R., Kirillova, F.M.: *High order necessary conditions for optimality*,- SIAM J Control, Vol 10, 1972, pp. 127–168.
[11] Stoer, J, and Bulirsch, J.: *Introduction to numerical analysis*, New York: Springer-Verlag, 1993.
[12] Wolfram, S. *The Mathematica Book, 5th ed.*Champaign, IL: Wolfram Media, 2003.
[13] Jeremić, B., Radulović, R., Obradović, A.*: Analysis of the brachistochronic motion of a variable mass nonholonomic mechanical system,* - Theoretical and Applied Mechanics, Vol 43, 2016, pp. 19-32.