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TIME OPTIMAL MOTIONS OF MECHANICAL SYSTEM WITH A PRESCRIBED TRAJECTORY

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Abstract: *The problem of time minimization of a holonomic scleronomic mechanical system, on a prescribed trajectory between two specified positions in configuration space, is solved. The generalized force with restricted coordinates is taken as the controlling force. Application of the Green theorem (the well-known Miele method in flight mechanics) has shown that in every instant, at least one control is at its boundary, and having controlling functions with interruptions. It is assumed that at least one generalized coordinate exists that is monotonous during the interval of movement. An algorithm for numerical computation is presented for assessing the boundary of the allowed area in the state space, thus, solving the problem of finding the optimal control as a function of time.*
Numerical integration is, therefore, carried out forward from the starting point, and backward from the ending point, by use of Runge-Kutta method. The mentioned procedure is illustrated in the example of time minimization for a manipulator, which has its tip moving in a straight line. The application of presented method simplifies solving of this type of problem in comparance to other methods, for instance, dynamic programming.
Generally, in the case of the manipulator device, it is necessary to numerically solve the apparent inverse kinematic problem only at end points. The method can be applied by using other criteria of optimality, and to other complex systems that have equations of state that also include equations describing the type of source, e.g. robots, having DC electric motors.

Key words: *optimal control, minimum time, mechanical system, prescribed trajectory, Miele method, manipulator*

1. INTRODUCTION. PROBLEM STATEMENT

In the optimal control problems of mechanical systems motion the most important place take the time minimization problem. Optimal control evaluation for systems with for larger number degrees of freedom by the application of mathematical theory of optimal control is very complex deal. Numerically unstable two-point boundary value problem has to be solved in Pontryagin's principle [7] approach. "Curse of dimensionality"[6] of dynamic programming makes this approach practically unacceptable in the case of large number of degrees of freedom. However, in the case of one degree of freedom, some algorithms of dynamic programming have been developed in literature [8,9,10] for various optimality criteria. In the case of time minimization simpler approach, in comparison with dynamic programming technique, has been presented in this paper. Contravariant form of differential equation of motion of holonomic scleronomic mechanical system with n degrees of freedom is given by the following expression:

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = a^{ij}\left(-\frac{\partial \Pi}{\partial q^j} + Q_j\right), \qquad (1)$$

$$i, j, k = 1,\ldots,n$$

where $q^i, i = 1,\ldots,n$ are generalized coordinates, $\Pi = \Pi(q^i)$ is potential energy, $a^{ij}(q^k), i, j, k = 1,\ldots,n$ are contravariant coordinates of metric tensor of configuration space, $Q_j, j = 1,\ldots,n$ are generalized forces, and $\Gamma^i_{jk} = \Gamma^i_{jk}(q^s), i, j, k, s = 1,\ldots,n$ are Christoffel's symbols of the second kind, given by the equations:

$$\Gamma^i_{jk} = \frac{1}{2}a^{is}\left(\frac{\partial a_{ks}}{\partial q^j} + \frac{\partial a_{sj}}{\partial q^k} - \frac{\partial a_{jk}}{\partial q^s}\right), \qquad (2)$$

$$i, j, k, s = 1,\ldots,n$$

where $a_{ij}(q^k), i, j, k = 1,\ldots,n$ represent covariant coordinates of metric tensor (Tensor notation [15] has been applied in the paper). Motion is controlled by the variable generalized forces with constrained coordinates:

$$|Q_j| \le C_j,\ C_j = const.,\ j = 1,\ldots,n \tag{3}$$

Solution of optimal control problem require determination controls $Q_j = Q_j(t)$,that satisfy (3), so the mechanical system, described by (1), in minimal travel time, from the given initial state:

$$t_o = 0,\ q^i(t_o) = q^{io}, \dot{q}^i(t_o) = 0,\ i = 1,\ldots,n \tag{4}$$

has to be transferred to the given final state:

$$t_o = ?,\ q^i(t_1) = q^{i1}, \dot{q}^i(t_1) = 0,\ i = 1,\ldots,n \tag{5}$$

Trajectory of the system in configuration space is described by the relation:

$$f^M(q^i) = 0,\ i = 1,\ldots,n,\ M = 1,\ldots,n-1 \tag{6}$$

where (4) and (5) have to satisfy (6).
Problem will be solved by the Miele method application [4], briefly described in the following chapter. Appropriate algorithm of postulated problem solution will be also presented. Minimum travel time problem for manipulator with determined tip trajectory will be solved by the mentioned procedure. Results of this considerations can be applied primary in the robotics. The most complete overview of developed methods, applied in the field of manipulators optimal control, is presented in the papers [1,13] and monograph [2]. Someone can find that algorithm given in this paper is applicable in problems of optimal control of nonredundant manipulator with determined tip trajectory.

2. BRIEF DESCRIPTION OF MIELE'S METHOD

At the beginning of fifties, before the maximum principle has appeared, Green's theorem has been applied by Angelo Miele in the solution of variational problems with inequality constraints [4]. Some problems of flight mechanics have been solved very efficiently by the direct application of former method [4,5,16].
Problem consists of determination of function $y = y(x)$, for which the functional:

$$H = \int_i^f [\phi(x,y) + \psi(x,y)y']dx = \int_i^f \phi dx + \psi dy \tag{7}$$

takes the extremum. Acceptable curves, between the initial point $I(x_I, y_I)$ and final point $F(x_F, y_F)$, have to lie in the domain of closed curve $\varepsilon(x,y) = 0$ (fig.1).
Curve domain is determined by the constraints of variables x, y, y'.

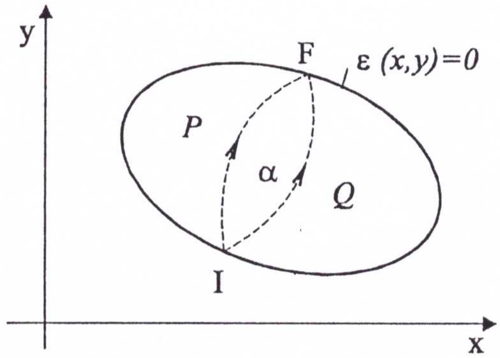


Fig.1.

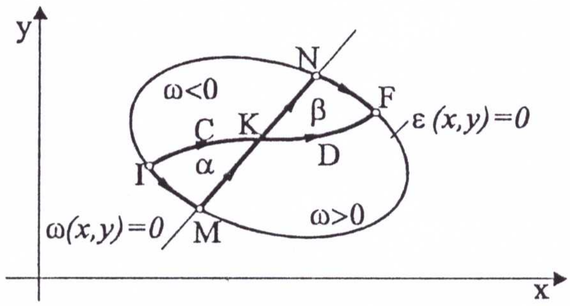


Fig.2

Functional comparation ΔH for different acceptable functions, that correspond to curves IF and IF:

$$\Delta H = \int_{IQF} \phi dx + \psi dy - \int_{IPF} \phi dx + \psi dy = \oint_{IQFPI} \phi dx + \psi dy \tag{8}$$

is determined. If $\Delta H > 0$, curve IF is "better" in the sence of optimality, when minimum value of H is required. If the condition $\Delta H > 0$ is satisfied for arbitrary chosen curve IQF, then the function $y = y(x)$, corresponding to curve IPF, is the solution of the postulated problem. If the both functions $\varphi(x,y)$, $\psi(x,y)$ and their first derivatives continuous in domain α , according to Green's theorem expression (8) is transformed to:

$$\Delta H = \iint_\alpha \left(\frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y}\right) dxdy = \iint_\alpha \omega(x,y) dxdy \tag{9}$$

Function $\omega(x, y)$ is called by Angelo Miele "fundamental function". Line $\omega(x, y) = 0$ in general case intersects line $\varepsilon(x, y) = 0$ (fig.2). Function according to curve IMNF is optimal in the sence of minimality of H, since $\Delta H > 0$ for arbitrarily chosen curve. Arbitrarily chosen curve ICKDF, that intersects curve $\omega(x, y) = 0$ only once, corresponds to function for which:

$$\Delta H = \int_{ICKDF} \phi dx + \psi dy - \int_{IMKNF} \phi dx + \psi dy = = -\iint_{\alpha} \omega dxdy + \iint_{\beta} \omega dxdy > 0 \quad (10)$$

is satisfied. In the case that $\omega(x, y) = 0$ and $\varepsilon(x, y) = 0$ do not intersect optimal curve lies on the boundary $\varepsilon(x, y) = 0$ with determined direction.

The advantage of this method is the fact that the absolute minimum is determined. The obstacle is very complex determination of acceptable region and application in the case of large number of variables. Mostly, acceptable regions can be find' at least one monotonous variable exists (time, variable rocket mass,etc.). Mentioned reasons makes this method very applicable in flight mechanics.

3. TIME MINIMIZATION PROBLEM

Let us propose that exist at least one generalized coordinate, monotonous in the interval of motion, as has been done in [8]. It can be, without loss of generality, coordinate q^1, that also satisfies $\dot{q}^1 \geq 0$.

System of equations (1) can be written in the form:

$$\dot{q}^i = y^i, \quad \dot{y}^i = \psi^i + a^{ij} Q_j, \quad i, j = 1, \ldots, n \quad (11)$$

where:

$$\psi^i = -\Gamma^i_{jk} y^j y^k - a^{ij} \frac{\partial \Pi}{\partial q^i}, \; i, j, k = 1, \ldots, n \quad (12)$$

Derivation of (6), according to (11), leads to relations, satisfied along the given trajectory by generalized velocities:

$$\frac{\partial f^M}{\partial q^k} y^k = 0, \; k = 1, \ldots, n, \quad M = 1, \ldots n-1 \quad (13)$$

and generalized forces:

$$\frac{\partial^2 f^M}{\partial q^k \partial q^i} y^i y^k + \frac{\partial f^M}{\partial q^k} \left(\psi^k + a^{ik} Q_j \right) = 0 \quad (14)$$

$i, k = 1, \ldots, n, \quad M = 1, \ldots n-1$

By the integration of the differential equation:

$$\ddot{q}^1 = \frac{d\left[\left(\dot{q}^1 \right)^2 \right]}{2\, dq^1} = \psi^1 + a^{1j} Q_j, \; j = 1, \ldots, n \quad (15)$$

the domain of acceptable function in phase plane (q^1, y^1) is obtained. Forward integration from the initial point $I(q^{10}, 0)$, where the controls Q_j are determined from the maximum of:

$$a^{1j} Q_j, \quad j = 1, \ldots, n \quad (16)$$

according (3),(6),(13) and (14), the part of the boundary domain is obtained. Another part (fig.3) of boundary is determined by the backward integration from the final point $F(q^{11}, 0)$, where the controls Q_j are determined from the minimum of (16). Lower bound IF corresponds to infinite time duration of motion.

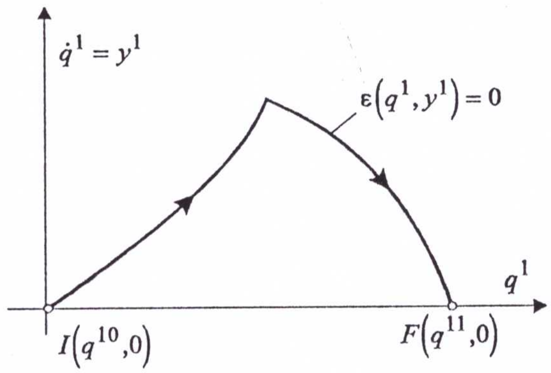


Fig.3.

One can write the functional:

$$H = \int_i^f dt = \int_i^f \frac{1}{y^1} dq^1 + 0 \cdot dy^1 \quad (17)$$

and determine fundamental function:

$$\omega\left(q^1, y^1\right) = \frac{\partial}{\partial q^1}(0) - \frac{\partial}{\partial y^1}\left(\frac{1}{y^1}\right) = \frac{1}{\left(y^1\right)^2} \quad (18)$$

According to:

$$\omega\left(q^1, y^1\right) > 0 \quad (19)$$

and previous considerations it can be concluded that optimal solution corresponds to the upper bound of acceptable region.

4. DESCRIPTION OF THE ALGORITHM

From the consideration in the previous chapter, algorithm of boundary determination can be explained, that represents numerical solution of optimal control problem.
System (11) is integrated from the initial point (4) and controls Q_j are determined from the maximum of (16) with equality constraints (14) and inequality constraints (3). According to linearity of Q_j in (14) and (16), solution is found at boundary (3). Maximum is determined by the comparison of 2n possible solutions. By the application of the Verner-Runge- Kutta numerical integration scheme generalized coordinates, velocities and forces (controls) in time have been evaluated.
Backward integration of system (11) from the final point (5) has been performed. According to autonomity of system (11) (time is not explicitly present) it follows $t_1 = 0$. The procedure of determination Q_j is almost the same as in the previous phase, with minimum value of (16).
It is necessary to find time instant t' of forward integration and time instant $-t''$ of backward integration. These instants are determined from the equality of values q^1 and y^1 , evaluated in forward and backward integration:

$$q^1(t') = q^1(-t''), \quad y^1(t') = y^1(-t''), \tag{20}$$

With the known t' and t'' the total minimum travel time $t_1 = t' + t''$ is evaluated.
Since the controls satisfy (14), and initial values (4) and final values (5) satisfy (6) and (13), relation (6) is satisfied in every time instant.
The extremely complex procedure of solution of system (6) with unknown $q^k, k = 2,3,\ldots,n$ is not necessary.
The final step of algorithm is forward integration of system (11) in $[t', t_1]$ with the initial values in the point t', where the controls Q_j are determined by the minimum of (16). Optimal controls and state variables (generalized coordinates and velocities) have been numerically evaluated in time. In that way problem is finally solved.

5. NUMERICAL EXAMPLE

Presented procedure can be illustrated on the minimum travel time problem of manipulator in horizontal plane motion [8], with two degrees of freedom:

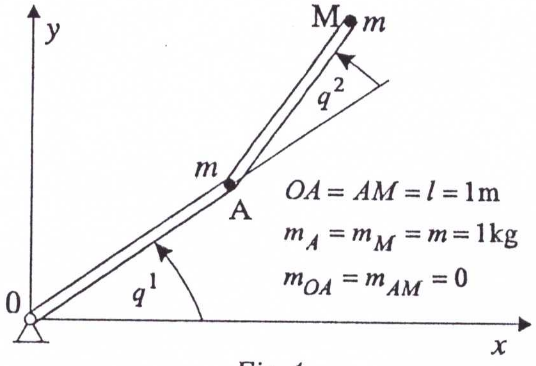


Fig.4

Tip M moves on straight line given by:

$$x_M + y_M - 2l = 0 \tag{21}$$

The expression (6) takes the following form:

$$\cos q^1 + \cos\left(q^1 + q^2\right) + \sin q^1 + \sin\left(q^1 + q^2\right) - 2 = 0 \tag{22}$$

Manipulator is stationary at the two end points, determined by:

$$x_M(t_o) = 0.75\,\text{m}, \quad x_M(t_1) = 0.5\,\text{m}, \tag{23}$$

Problem has been solved for two cases of actuator torque limits:

$$\begin{aligned} &\text{a)}\ C_1 = 3.\ \text{Nm} \quad C_2 = 2.\ \text{Nm} \\ &\text{b)}\ C_1 = 3.\ \text{Nm} \quad C_2 = 0.65\ \text{Nm} \end{aligned} \tag{24}$$

It is also necessary to determine $a_{ij}, i, j = 1,2$:

$$\left[a_{ij}\right] = ml^2 \begin{bmatrix} 3 + 2\cos q^2 & 1 + \cos q^2 \\ 1 + \cos q^2 & 1 \end{bmatrix} \tag{25}$$

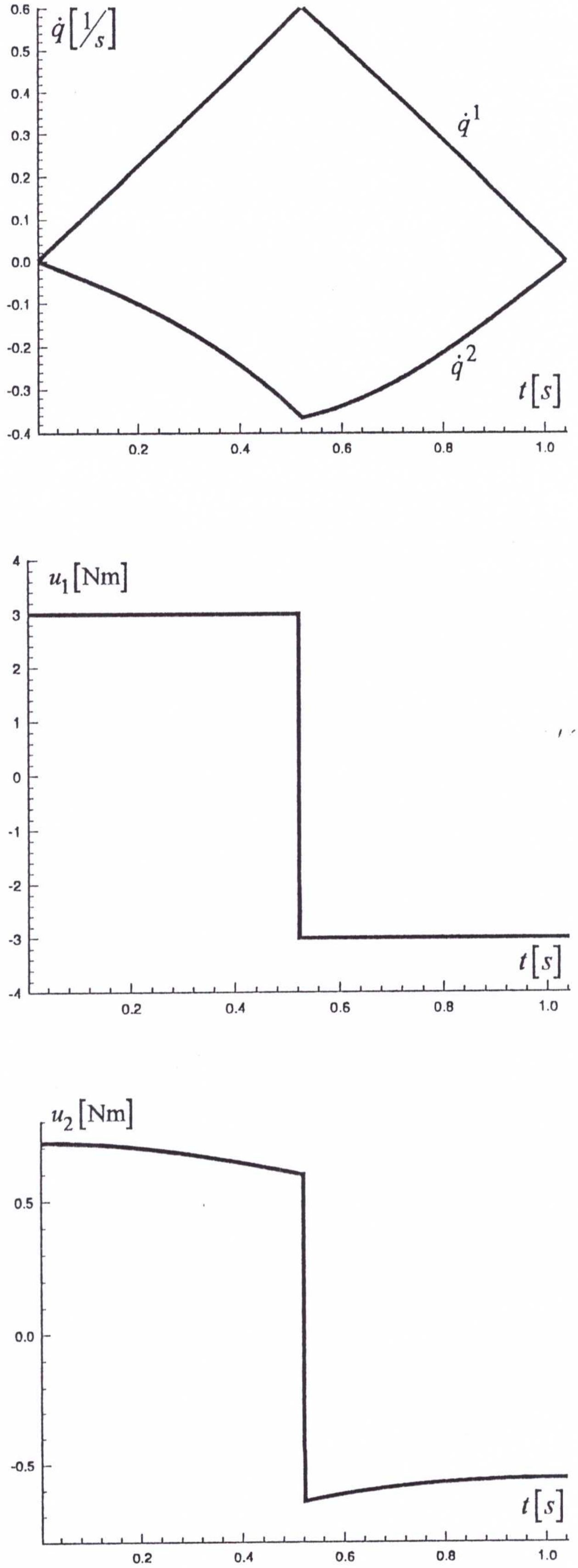
potential energy:

$$\Pi = 0 \tag{26}$$

and solve the inverse kinematic problem at the end points:

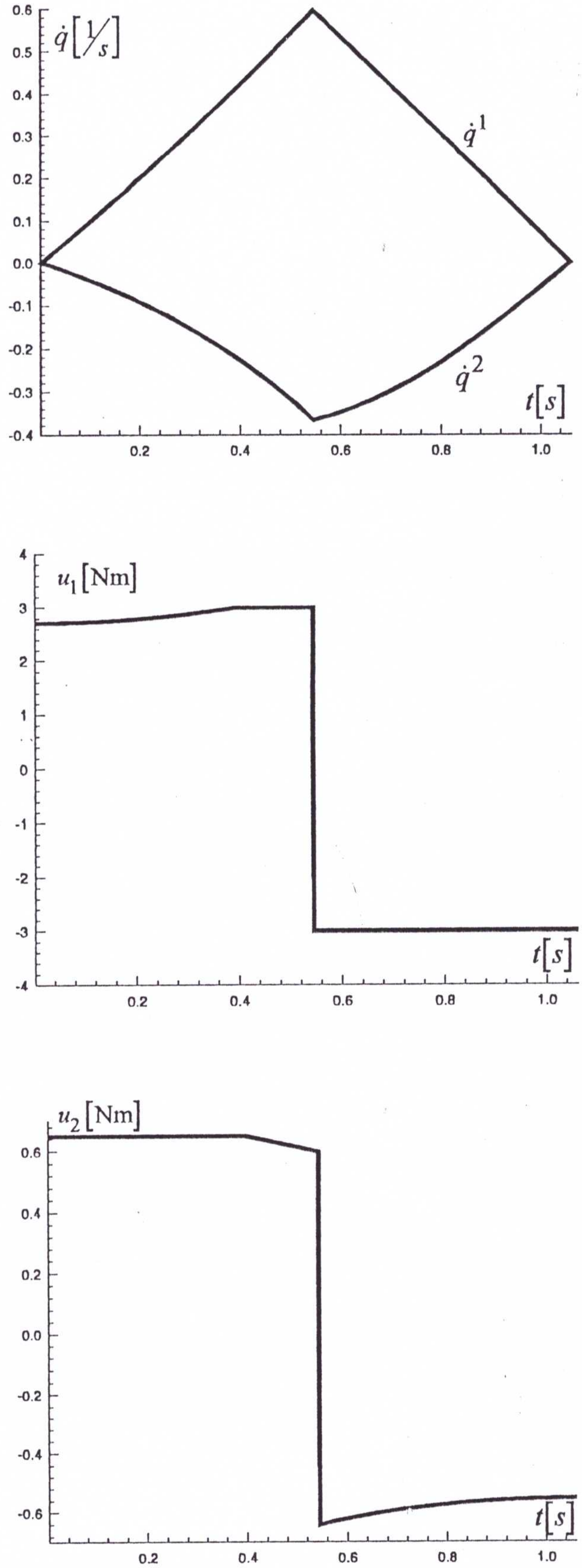
$$\begin{aligned} q^1(t_o) &= 15.8279^o \quad q^2(t_o) = 86.4167^o \\ q^1(t_1) &= 33.8038^o \quad q^2(t_1) = 75.5225^o \end{aligned} \tag{27}$$

where only one of two possible manipulator configurations is chosen.
According to (25) and (26) system of equations (11) is obtained. By the application of the algorithm from chapter 4 optimal controls and corresponding trajectory have been determined. Some relevant results are presented on fig.5a and fig.5b.



$\dot{q}\left[\frac{1}{s}\right]$
$\dot{q}^1$
$\dot{q}^2$
$t[s]$
u_1 [Nm]
u_2 [Nm]
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
0.2
0.4
0.6
0.8
1.0

Fig 5a.



$\dot{q}\left[\frac{1}{s}\right]$
$\dot{q}^1$
$\dot{q}^2$
$t[s]$
u_1 [Nm]
u_2 [Nm]
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0.2
0.4
0.6
0.8
1.0

Fig 5b

In the case a) control Q_1 is on the boundary during the motion, with break at time instant $t = 0.521s$. Minimum travel time is $t_1 = 1.042$.
In the case b) control Q_2 is on the upper bound until time instant $t = 0.396s$ is reached. After that control Q_1 is on the upper bound until time instant $t = 0.543s$ is reached. In the remaining part of motion control Q_1 is on the lower bound. Minimum travel time is $t_1 = 1.069s$.
Results from the case a) are identical with the results of [8], while results from the case b) are not considered.

6. CONCLUSION

Present method of travel time minimization gives absolute minimum, since evaluations are not based on necessary condition of optimality, which is the case of the most other methods. Present algorithm has simpler structure than the technique of dynamic programming (there are not recursion, large computer storage, etc.).
By the application of the present method, very complex inverse kinematic problem with non unique solution, in the case of manipulators, is sufficient to be solved only at the end points. It is not necessary to parametrize the trajectory as it is usual in similar procedures [11,12].
Present algorithm can be also used in the case of complex system models that take into account equations of drives. Let us consider equations describing a robot with electric DC drives [1]:

$$a_{ij}^{*}\ddot{q}^{j} + b_i^{*} = Q_i^{*}, \quad i, j = 1,\dots,n \tag{28}$$

where is:

$$a_{ij}^{*} = a_{ij}^{*}\left(q^{k}\right),\ b_i^{*} = b_i^{*}\left(q^{k}, \dot{q}^{s}\right), \quad i, j, k, s = 1,\dots,n \tag{29}$$

and control constraints have the form:

$$\left|Q_i^{*}\right| \le C_i^{*},\ C_i^{*} = const.,\ i = 1,\dots,n \tag{30}$$

To obtain the equations (28) and constraints (30) it is necessary to know parameters of DC drives: moments of inertia of the rotating parts, gear ratio, maximum voltages, etc. In that case it is necessary to take into account equations (30) and (28) instead of (3) and (11), respectively.
Apart of the time minimization problem, method Miele can be applied on the other optimality criteria in the case of mechanical systems with one degree of freedom [3].

REFERENCES

[1] Chernousko,F.L., Optimal Control of Robotic Systems, Zbornik radova 21. Jugoslovenskog kongresa teorijske i primenjene mehanike, Predavanja po pozivu, Niš, 1995.
[2] Chernousko,F.L.,Bolotnik,N.N.,Gradetsky,V.G., Manipulation Robots, CRC Press, Boca Raton, Florida,USA, 1994.
[3] Obradović,A., Primena Grinove teoreme (metod Miele) u optimalnom upravljanju kretanjem sistema sa jednim stepenom slobode, Zbornik radova 21. Jugoslovenskog kongresa teorijske i primenjene mehanike, Sveska A, Niš, 1995.
[4] Миеле,А., Определение экстремумов криволинейных интегралов по теореме Грина, Методи оптимизации с приложениями к механике космического полета, Наука, Москва, 1965.
[5] Летов,А.М., Динамика полета и управление, Наука, Москва, 1969.
[6] Sage,A.P.,White, C.C., Optimum Systems Control, Prentice-Hall, Englewood, 1977.
[7] Понтрягин,Л.С.,Болтянский,В.Г.,Гамкрелиде, Р.В.,Мищенко, Е.Ф., Математическая теория оптимальных процессов, Наука, Москва, 1983.
[8] Singh,S.,Leu,M.C., Optimal Trajectory Generation for Robotic Manipulators Using Dynamic Programming, ASME Journal of Dynamic System,Measurement and Control, Vol.109, June 1987.
[9] Shin,K.G.,McKay,N.D., A Dynamic Programming Approach to Trajectory Planning of Robotic Manipulators, IEEE Transactions on AC, Vol.AC-31, No 6, June 1986.
[10] Vukobratović,M.,Kirćanski,M., A Method for Optimal Synthesis of Manipulation robot Trajectories, ASME Journal of Dynamic System,Measurement and Control, Vol.104, June 1982.
[11] Pašić,M., Time Optimal Motion of the Tip of Manipulators, Facta Universitatis, Series Mechanics,Automatic Control and Robotics, Vol.1,No2, 1992.
[12] Bobrow,J.E., Dubovsky,S.,Gibson,J.S., Time Optimal Control of Robotic Manipulators along specified paths, The International Journal of Robotics Research, Vol.4,No3, 1985.
[13] Болотник, Н.Н.,Черноусько, Ф.Л., Оптимизация управления манипуляционными роботами, Известия АН СССР, Техническая кибернетика, Но 1, 1990.
[14] Лурье, А.И., Аналитическая механика, Гос.изд.физ.-мат. литератури, Москва, 1961.
[15] Anđelić, T.P., Tenzorski račun, Naučna knjiga, Beograd, 1987.
[16] Leitmann,G., An Introduction to Optimal Control, McGraw-Hill, New York, 1966.

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