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OPTIMIZATION OF SHOCK ABSORBERS WITH NON-LINEAR CHARACTERISTIC

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Abstract: The authors have discussed the problem of optimal attenuation of a mechanical system in a general case in the paper "Optimal Control and Attenuation of the Oscillating Systems" (Proc. of XIV ICMHW '96) . Classification of basic optimal control tasks and their mathematical modeling in the form suitable for application of methods of contemporary optimal control theory have been discussed. A method, based upon previous considerations, was used by the authors for further examination. In this paper, the problem of optimization of shock absorbers with non-linear characteristic has been solved. In order to protect the attenuated body from the consequences of external influences (shock), two basic problems of optimal attenuation have been discussed. Maximum relative displacement and maximum force acting on attenuated body have been represented in the form of a functional depending on characteristics of absorber as a control function. It can be seen from the solution of these problems that absorbers with the damping force of the second order and linear restitution force have some special properties.

Keywords: Maximum Principle, Optimization, Shock Absorber, Non-Linear

INTRODUCTION

Among problems of oscillating motion optimization a problem of optimal attenuation takes a special place. Impact effects and great accelerations can damage the cargo during the transport or disturb the work of accurate instruments and devices. Limitations of real conditions dictate the choise of absorbers for protection from mentioned effects, therefore optimal solution of the problem is often necessary. A large bibliography in monographs (Sevin and Pilkey, 1971., Chernousko et al., 1980.) confirms that the development of procedure of optimal theory allows contemporary researches of attenuation problems to be direct that way. The great number of these works considers particular attenuation problem. Current computer techniques move researches toward more complex problems; therefore theoretical consideration of more general types of system is required. Attenuation of general model of mechanical system was considered, basic optimal problem classification was made methods of their solution were pointed out in previous researches (Vuković and Obradović, 1996.b) of authors of this paper. In that sense the aim of this work is to apply completely the method based of maximum principle (Leitmann, 1966., Sage and White, 1977. , Pontryagin et al., 1983.) to the problem with known solution. An absorber characteristic determination problem for protection from impact effects is solved. This problem was solved separately by a great number of authors (Sevin and Pilkey, 1971., Chernousko et al., 1980., Bolotnik, 1977.) and was used as illustration of various problems.

1. IMPACT EFFECTS. ABSORBER CHARACTERISTIC

Let the considered system be consisted of body O with mass M and body A with mass m (Fig.1). Position of the system is determined by absolute coordinate z and relative coordinate x , representing position of body A in regard to body O. The bodies are mutually connected by absorber in such way that its force Q (absorber characteristic) depends only on relative state $(x, \dot{x})$ of body A in regard to body O, i.e. :

$$Q = Q(x, \dot{x}) \quad (1.1)$$

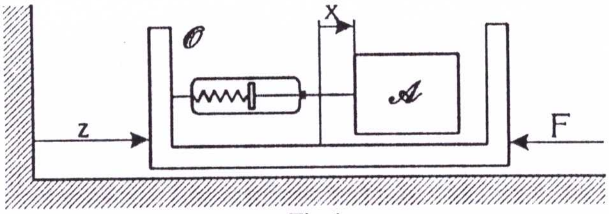

Fig.1

Let basic body O be exposed to action of external force F acting opposite to the system motion, during the motion. Kinetic energy of considered system has form :

$$T=\frac{1}{2}M\dot{z}^2+\frac{1}{2}m(\dot{z}+\dot{x})^2 \qquad (1.2)$$

so those differential equations of motion are :

$$(M+m)\ddot{z}+m\ddot{x}=-F$$
$$m(\ddot{z}+\ddot{x})=Q \qquad (1.3)$$

Behavior of the system initiated by action of impact force will be considered

$$F(t)=\begin{cases}0, t<t^* \\ F, t\in[t^*,t^*+\tau] \quad (\tau\to 0) \\ 0, t>t^*+\tau\end{cases} \qquad (1.4)$$

Impact force impulse has finite value in interval $[t^*,t^*+\tau]$ $(\tau\to 0)$, i.e. :

$$I=\int_{t^*}^{t^*+\tau}F\,dt\ . \qquad (1.5)$$

The force Q of the absorber is limited and finite therefore its impulse in same interval equals zero. On the basis of these characteristic of forces F and Q, integral of equations (1.3) in interval $[t^*,t^*+\tau]$ gives:

$$(M+m)\dot{z}(t^*+\tau)+m\dot{x}(t^*+\tau)-$$
$$-(M+m)\dot{z}(t^*)-m\dot{x}(t^*)=-I,$$
$$\dot{z}(t^*+\tau)+\dot{x}(t^*+\tau)-\dot{z}(t^*)-\dot{x}(t^*)=0,$$
$$z(t^*+\tau)-z(t^*)=0, x(t^*+\tau)-x(t^*)=0. \qquad (1.6)$$

In the moment of impact t^* let body A be in relative equilibrium in regard to basic body O $(\dot{x}(t^*)=0, x(t^*)=0)$ than on the basis of (1.6) it is :

$$\dot{x}(t^*+\tau)=\frac{I}{M}\ ,\ x(t^*+\tau)=0. \qquad (1.7)$$

During the further motion after the impact $(t\geq t^*+\tau)$ in absence of external forces (F=0) , from (1.6) integral of system momentum follows :

$$(M+m)\dot{z}+m\dot{x}=(M+m)\dot{z}(t^*+\tau)+m\dot{x}(t^*+\tau)=C_0\ ,$$
$$\forall t\geq t^*+\tau, \qquad (1.8)$$

therefore kinetic energy (1.2) can be transformed into form :

$$T=C+\frac{1}{2}\mu\dot{x}^2\ , \qquad (1.9)$$

where :

$$C=\frac{C_0{}^2}{2(M+m)}\ ,\quad \mu=\frac{Mm}{M+m}\ . \qquad (1.10)$$

On the basis of (1.9) or (1.3) the following equation is achieved :

$$\mu\ddot{x}=Q(x,\dot{x}) \qquad \forall t\geq t^*+\tau \qquad (1.11)$$

which describes relative motion of body A due to action of absorber after the impact. Due to autonomy of equation (1.11) for further motion $(t\geq t^*+\tau)$ it can be introduced initial moment $t_0=t^*+\tau=0$ and considering (1.7) initial conditions :

$$t_0=0\ ,\ x(0)=0\ ,\ \dot{x}(0)=\frac{I}{M}=V_0\ . \qquad (1.12)$$

Equation (1.11) with conditions (1.12) shows that behavior of body A after the impact depends only on chosen absorber characteristic. For attenuation of relative motion of body A it is desirable that absorber has characteristic :

$$Q(x,\dot{x})=Q_e(x)+Q_w(\dot{x})\ . \qquad (1.13)$$

In practice characteristic in form :

$$Q=-c|x|^k\, sign\, x-b|\dot{x}|^l\, sign\,\dot{x} \qquad (1.14)$$

is of great significance, where stiffness coefficient c and damping constant b are positive constants and k and l are cardinal numbers. By introducing potential energy Π and dissipative function Φ :

$$\Pi=\frac{1}{k+1}c|x|^{k+1}\ ,\ \Phi=\frac{1}{l+1}b|\dot{x}|^{l+1}\ , \qquad (1.15)$$

absorber characteristic (1.14) can be expressed as :

$$Q=-\frac{\partial\Pi}{\partial x}-\frac{\partial\Phi}{\partial\dot{x}} \qquad (1.16)$$

Expression (1.16) leads to known law of mechanical energy dissipation :

$$\frac{d}{dt}(T+\Pi)=-(l+1)\Phi \qquad (1.17)$$

that is, considering (1.9) and (1.15) :

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{x}^2+\frac{1}{k+1}c|x|^{k+1}\right)=-b|\dot{x}|^{l+1}\leq 0\ . \qquad (1.18)$$

2. BASIC PROBLEMS OF OPTIMAL ATTENUATION. CRITERION OF OPTIMALITY

The size of considered system and its sensitivity to acceleration impose maximal relative motion of body A and its maximal absolute acceleration to be basic variables to consider in optimal attenuation problem. Both variables depend on absorber characteristic Q and maximal absolute accelerations to maximal force Q; therefore functionals that should be considered are :

$$J_1(Q)=\max_t|x(t)|\ ,\quad J_2(Q)=\max_t|Q[x(t),\dot{x}(t)]|. \qquad (2.1)$$

Functions $x(t)$ and $\dot{x}(t)$ are continuous, therefore functionals have local maximum character. Therefore it should be considered only those with the greatest values. Let $t_i\ (i=1,2,3,\dots)$ be moments in which $x(t)$ have local extremes , i.e.: $\dot{x}(t_i)=0$, $t_1<t_2<t_3<\ldots$, then, on the basis dissipation law (1.18), it follows :

$$|x(t_1)|>|x(t_2)|>|x(t_3)|>\ldots \quad (2.2)$$

For arbitrary chosen $t'\in[0,t_1]$ due to extreme $x(t_1)$ it always exist $t''\in[0,t_1]$ such that :

$$|x(t')|=|x(t'')| \quad , \quad (2.3)$$

therefore due to dissipation :

$$|\dot{x}(t')|\geq|\dot{x}(t'')| \quad . \quad (2.4)$$

Considering to (1.14) and expressions (2.3) and (2.4)

$$|Q[x(t'),\dot{x}(t')]|>|Q[x(t''),\dot{x}(t'')]| \quad (2.5)$$

that is , due to arbitrary variable $t'\in[0,t_1]$:

$$\max_{t\in[0,t_1]}|Q[x(t),\dot{x}(t)]|>\max_{t\in[t_1,\infty)}|Q[x(t),\dot{x}(t)]| \quad (2.6)$$

On the basis of (2.3) and (2.6) it can be concluded that body A achieves maximal relative displacement in t_1 , and maximal acceleration somewhere in interval $[0,t_1]$, therefore it is sufficient, optimal attenuation problem to be consider only in interval $[0,t_1]$. In that case functionals (2.1) can be expressed in following manner :

$$J_1(Q)=|x(t_1)| \quad , \quad J_2(Q)=\max_{t\in[0,t_1]}|Q[x(t),\dot{x}(t)]| \quad , (2.7)$$

where Q represents control function for motion of the system described by equation :

$$\mu\ddot{x}=Q \qquad \forall t\in[0,t_1] \quad (2.8)$$

with conditions on the ends of interval $[0,t_1]$:

$$x(0)=0 \ , \ \dot{x}(0)=V_0 \ , \ \dot{x}(t_1)=0 \quad . \quad (2.9)$$

Let $Q\in U$, where U is limited or unlimited set of allowable controls, then considering to functionals (2.7) two basic problems of optimal control can be defined :

1. Minimization of absorber operation. Let the absolute acceleration of attenuated body A be limited, what considering to (1.3) , limits set U of allowable controls in form :

$$U:|Q|\leq K \quad (2.10)$$

where K is known constant. Absorber operation is determined by displacement x of body A in such manner that its maximal value is $x(t_1)$. The problem of optimal minimization of absorber operation is to determine from (2.10) among allowable controls optimal controls Q^o for which maximal displacement of body A would be the smallest, that is :

$$J_1(Q^o)=\min_{|Q|\leq K}J_1(Q)=\min_{|Q|\leq K}|x(t_1)| \quad . \quad (2.11)$$

2. Minimization of acceleration. Let the range U of allowable controls be open set and let the maximal relative displacement of body A (absorber operation) be limited :

$$|x(t_1)|\leq d \ , \quad (2.12)$$

where d is known constant. If the minimization of absolute acceleration of body A is required, then optimality criterion has form :

$$J_2(Q^o)=\min_{Q\in U}J_2(Q)=\min_{Q\in U}\max_{t\in[0,t_1]}|Q| \quad . \quad (2.13)$$

These formulations of optimality problem refer to direct application of methods based on mathematical theory of optimal control (Leitmann, 1966., Sage and White, 1977. , Pontryagin et al., 1983.)

3. THE MAXIMUM PRINCIPLE. DETERMINATION OF OPTIMAL CHARACTERISTIC OF ABSORBER

In accordance with the maximum principle(Pontryagin et al., 1983.) the state of controlled system is considered in phase space (space of state). By introducing phase coordinate (state variables) x and y, equation (2.8) can be replaced with system of equations:

$$\dot{x}=y \ , \ \dot{y}=\frac{Q}{\mu} \ , \quad \forall t\in[0,t_1] \ , \quad (3.1)$$

and conditions (2.9) with conditions :

$$x(0)=0 \ , \ y(0)=V_0 \ , \ y(t_1)=0 \quad . \quad (3.2)$$

On the basis of directions of coordinate x and impulse I (Fig.1) follows that $V_0>0$ and that in interval $[0,t_1]$:

$$x\geq 0, y\geq 0, \ \forall t\in[0,t_1] \ , \quad (3.3)$$

therefore characteristic (1.14) becomes in form :

$$Q=-c\,x^k-b\,y^l \ , \ \forall t\in[0,t_1] \quad . \quad (3.4)$$

In general the maximum principle does not solve directly the problem of synthesis of optimal control Q^o in form (3.4), but gives the solution $Q^o=Q^o(t)$,which represents first step in solutions of previous problems.

Solution of problem 1. Considering (3.1),(3.2) and (3.3) functional $J_1(Q)$ can be transformed in such way that optimality criterion becomes in form :

$$J_1(Q^o)=\min_{|Q|\leq K}\int_0^{t_1}y\,dt \quad . \quad (3.5)$$

Pontryagin's function (Pontryagin et al., 1983.) of optimal problem (3.1),(3.2) and (3.5) is :

$$H=\psi_0 y+\psi_1 y+\psi_2\frac{Q}{\mu} \quad (3.6)$$

where ψ_0 , ψ_1 , ψ_2 can not equal zero et the same time and represent solutions of system of equations :

$$\psi_0=const\leq 0 \ , \ \dot{\psi}_1=0 \ , \ \dot{\psi}_2=-\psi_0-\psi_1 \ , \quad (3.7)$$

with transversality condition

$$\psi_1(t_1)=0 \quad . \quad (3.8)$$

Due to autonomy of equation (3.1) and indeterminate of t_1 optimality criterion has form :

$$(H)_{Q^o}=\sup_{|Q|\leq K}H=0 \quad , \quad \forall t\in[0,t_1] \quad . \quad (3.9)$$

Equation (3.1) and function (3.6) are linearly dependent on variable Q which refer to the possibility of existence of singular optimal control (Obradović, 1995.). However, on the basis of necessary conditions for singular controls :

$$\frac{\partial H}{\partial Q} = 0 \ , \ |Q| < K \ , \tag{3.10}$$

and on the basis of equations (3.7) and condition (3.8) follows $\psi_0 = 0 \ , \ \psi_1 = 0 \ , \ \psi_2 = 0 \ , \ \forall t \in [0, t_1]$,what is contradictory to the maximum principle, therefore optimal control should be searched on the boundary of set (2.10),i.e.

$|Q^o(t)| = K$. In that case condition (3.9) is satisfied when :

$$Q^o = K \, sign \, \psi_2 \ , \ \forall t \in [0, t_1] \ . \tag{3.11}$$

From equations (3.7), on the basis (3.8) and (3.9),

$$\psi_0 < 0, \ \psi_2 = \psi_0 (t_1 - t) \ , \ \forall t \in [0, t_1] \ , \tag{3.12}$$

therefore optimal solution is :

$$Q^o(t) = -K \ , \ \forall t \in [0, t_1] \ . \tag{3.13}$$

By including these values in equations (3.1), on the basis of (3.2), optimal state equations of attenuated body A in interval $[0, t_1]$ are achieved :

$$y^o = -\frac{K}{\mu} t + V_0 \ , \ x^o = -\frac{K}{2\mu} t^2 + V_0 t \ , \ t \in [0, t_1] , \tag{3.14}$$

follows :

$$x_1^o = x^o(t_1) = \frac{\mu V_0^{\,2}}{2K} \ , \ t_1 = \frac{\mu V_0}{K} \ . \tag{3.15}$$

Solution of problem of optimal characteristic synthesis is to determine, among absorbers with characteristic (1.14), that is (3.4), those which realize force with constant intensity K in interval $[0, t_1]$. In that case :

$$\frac{dQ^o}{dt} \equiv 0 \ , \ \forall t \in [0, t_1] \ , \tag{3.16}$$

and on the basis of (3.1) and (3.4) follows :

$$ckx^{k-1} + \frac{K}{\mu} bly^{l-2} \equiv 0 \ , \ \forall t \in [0, t_1] \ . \tag{3.17}$$

On the basis of equations (3.15) it is shown that this identity has sense only if :

$$k = 1 \ , \ l = 2 \ , \tag{3.18}$$

therefore optimal characteristic of absorber has form :

$$Q^o = -c^o x - b^o \dot{x}^2 sign \, \dot{x} \ . \tag{3.19}$$

From the conditions on the ends of interval $[0, t_1]$ optimally coefficient are determined :

$$c^o = \frac{2K^2}{\mu V_0^{\,2}} \ , \ b^o = \frac{K}{V_o^{\,2}} \ . \tag{3.20}$$

In that manner, absorber, with optimal characteristic (3.19) and (3.20), realize in interval $[0, t_1]$ force of maximal constant intensity. During the further motion ($t > t_1$) , force (3.19) is variable, and its intensity is smaller then the maximal one due to dissipation (2.6), i.e. :

$$|Q^o| = \begin{cases} K, t \in [0, t_1] \\ |Q| < K, t > t_1 \end{cases} \ . \tag{3.21}$$

Relative motion of body A which is determined by (3.14) in interval $[0, t_1]$, during the further motion $(t > t_1)$ is oscillating (non-linear oscillations) with regressive amplitude, therefore it is $x(t) < x_1^o \ , \forall t > t_1$. In the Fig.2 diagrams $x^o(t)$ and $Q^o(t)$ achieved as numerical integration of (3.1) with optimal characteristics (3.19) and (3.20) are shown

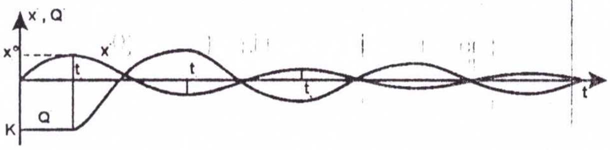

Fig.2

<u>Solution of problem 2</u>. Let :

$$\max_{t \in [0, t_1]} |Q| = D \tag{3.22}$$

where D is undetermined constant from open set (D > 0), then optimality criterion (2.13) has form :

$$J_2(Q^o) = \min_{Q \in U} D \ . \tag{3.23}$$

Range of allowable controls U , considering to (3.22) , can be defined in the following manner :

$$U \ : \ |Q| \le D \ , \tag{3.24}$$

so that criterion (3.23) minimizes the boundary of allowable controls range. By introducing auxiliary variable ξ and auxiliary control u in such manner that it is :

$$\dot{\xi} = u \ , \ \xi(0) = 0 \ , \ \xi(t_1) = D \ , \tag{3.25}$$

criterion (3.23) can be transformed into form :

$$J_2(Q^o) = \min_{Q, u} \int_0^{t_1} u \, dt \ . \tag{3.26}$$

and the system of equations (3.1) can be expanded on :

$$\dot{x} = y \ , \ \dot{y} = \frac{Q}{\mu} \ , \ \dot{\xi} = u \ , \ \dot{D} = 0 \ , \tag{3.27}$$

with conditions on interval $[0, t_1]$ ends :

$$x(0) = 0 \ , \ y(0) = V_0 \ , \ \xi(0) = 0 \ ;$$

$$x(t_1) \le d \ , \ y(t_1) = 0 \ , \ \xi(t_1) - D(t_1) = 0 \ . \tag{3.28}$$

Pontryagin's function of problem (3.26),(3.27) and (3.28):

$$H = \psi_0 u + \psi_1 y + \psi_2 \frac{Q}{\mu} + \psi_3 u + \psi_4 0 \ , \tag{3.29}$$

where $\psi_i \ (i = 0,1,2,3,4)$, which can not all equal zero at the same time, represent the solution of equations :

$$\psi_0 = const \le 0, \dot{\psi}_1 = 0, \dot{\psi}_2 = -\psi_1, \dot{\psi}_3 = 0, \dot{\psi}_4 = 0 \ , \tag{3.30}$$

with transversality conditions :

$$\psi_4(0) = 0, \psi_3(t_1) + \psi_4(t_1) = 0,$$

$$\psi_1(t_1) = \begin{cases} 0, x(t_1) < d \\ C_1, x(t_1) = d \end{cases} , \tag{3.31}$$

where C_1 is arbitrary constant. Auxiliary control u belongs to the open set of allowable controls, therefore corresponding necessary condition of function H extreme is :

$$\frac{\partial H}{\partial u} = 0 \ \Rightarrow \ \psi_0 + \psi_3 = 0 \ . \tag{3.32}$$

From (3.30), (3.31) and (3.32) it is achieved :

$\psi_0 = 0$, $\psi_3 = 0$, $\psi_4 = 0$, $\forall t \in [0, t_1]$. (3.33)

If we presume that $x(t_1) < d$, then, on the basis of (3.31), (3.32) and condition $H = 0$, $\forall t \in [0, t_1]$, it follows $\psi_1 = 0$, $\psi_2 = 0$, which is along with (3.33) contradict to maximum principle; therefore presumption is not correct and :

$$x(t_1) < d \quad (3.34)$$

The presumption that there is singular optimal control ($|Q^o| < D$) leads to the same contradiction, therefore optimality criterion is satisfied only when :

$$Q^o = D \, sign \, \psi_2 \quad , \forall t \in [0, t_1] \; . \quad (3.35)$$

From (3.30) with conditions (3.31), (3.33) and $H = 0$, $\forall t \in [0, t_1]$, it is achieved :

$$\psi_1 = C_1 < 0 \; , \; \psi_2 = C_1(t_1 - t) \, , \forall t \in [0, t_1] \; , \quad (3.36)$$

On the basis of (3.36) it is :

$$Q^o = -D \; , \quad \forall t \in [0, t_1] \; . \quad (3.37)$$

The optimal movement of body A :

$$y^o = -\frac{1}{\mu} Dt + V_0 \, , x^o = -\frac{1}{2\mu} Dt^2 + V_0 t \, , t \in [0, t_1], \quad (3.38)$$

corresponds to solution (3.37) on the basis of (3.27), (3.28). From (3.38), by using conditions $x(t_1) = d$, $y(t_1) = 0$, it is :

$$D^o = \frac{\mu V_0^2}{2d} \quad , \; t_1 = \frac{2d}{V_0} \; . \quad (3.39)$$

The same consideration used in solution of the problem 1 gives the optimal characteristic :

$$Q^o = -c^o x - b^o \dot{x}^2 sign \, \dot{x} \quad (3.40)$$

where :

$$c^o = \frac{\mu V_0^2}{2d} \quad , \; b^o = \frac{\mu}{2d} \; . \quad (3.41)$$

In Fig.3 corresponding diagrams $x^o(t)$ and $Q^o(t)$ are shown:

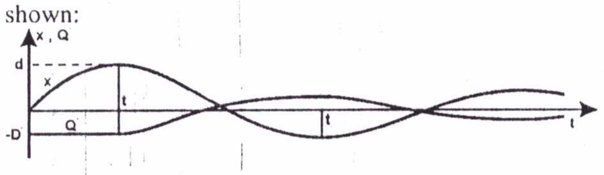

Fig.3

In considered problems if basic body O stops after the impact or if the influence of body A on the motion of basic body O is negligible ($\frac{m}{M} << 1$), then in achieved solutions following replacement can be made : $\mu = m$.

CONCLUSION

In problem 1 it is imposed limitation for the force to be ($|Q| \leq K$) and in problem 2 limitation for maximum displacement to be ($|x(t_1)| \leq d$). If in both problems both of limitation were imposed at the same time the solution of optimal problem would have sense only if $Kd \geq \frac{\mu V_0^2}{2}$. Solution of mentioned problem implies that optimal characteristic determination problem is to be considered in interval $[0, t_1]$, i.e. starting from the moment of impact t_0 till the t_1 when the displacement of absorber achieves maximum for the first time. Optimal force in that interval is constant what can be realized only by absorber whose characteristic is linear function of displacement and square function of velocity, with optimal values of corresponding coefficients. Absorbers with different characteristic will always more or less give more or less give more negative solution than achieved optimal solution. If absorber characteristic has predetermined form (k and l in (1.14) are known) than for the solution of optimal problems, optimal coefficients are to be determined. For solution of such problems method of optimal control with constant parameters (Vuković and Obradović, 1996. a) can be applied.

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