NUMERICAL SOLUTION TO OPTIMAL CONTROL PROBLEMS AND PARAMETER OPTIMIZATION OF MECHANICAL SYSTEMS

Zoran Marković[**], *Aleksandar Obradović*[*], *Josif Vuković*[*]
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1. INTRODUCTION- STATING OF THE PROBLEM

Application of contemporary mathematical theory on optimal control over mechanical and other systems is limited by calculating possibilities of optimal controls. Nowadays, there are no general algorithms with corresponding programmes for solution of this type limitation problems, so every solved assignment of this kind deserves attention. Consequently, in current professional literature there is a considerable number of works on solving actual assignments of optimal control.

This paper is concerned with holonomic scleronomic mechanical systems the motion of which, subjected to action of potential and non-potential forces, can be described by means of Lagrange's equation of the second type in countervariant form:

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = a^{ij}\left(-\frac{\partial \Pi}{\partial q^j} + Q^N_j\right),$$

$$i, j, k = 1, 2, \ldots, n, \qquad (1)$$

where q^i, $i=1, \ldots, n$ are generalized coordinates, $\Pi = \Pi(q^i, c_\alpha)$, $i=1, \ldots, n$, $\alpha=1, \ldots, s$ is potential energy of the system which, in this problem also depends on constant parameters c_α, $a^{ij}=a^{ij}(q^k, c_\alpha)$, $i,j,k=1, \ldots, n$, $\alpha=1, \ldots, s$ countervariant coordinate of metric tensor, $Q^N_j = Q^N_j(q^i, \dot{q}^k, u_\beta, c_\alpha)$, $i,j,k=1, \ldots, n$, $\alpha=1, \ldots, s$, $\beta=1, \ldots, m$ are non-potential generalized forces, u_β is control vector and Γ^i_{jk} are Christoffel's symbols of the second type, given in the following expression:

$$\Gamma^i_{jk} = \frac{1}{2}a^{il}\left(\frac{\partial a_{kl}}{\partial q^j} + \frac{\partial a_{lj}}{\partial q^k} - \frac{\partial a_{jk}}{\partial q^l}\right),$$

$$i, j, k, l = 1, 2, \ldots, n. \qquad (2)$$

where $a_{ij}=a_{ij}(q^k, c_\alpha)$, $i,j,k=1, \ldots, n$, $\alpha=1, \ldots, s$, are covariant coordinates of metric tensor.

For the purpose of applying Pontryagin's principle of the maximum, it is necessary to substitute the system consisting of n differential equations of the second order (1) with the 2n differential equations of the first order. Formally, in order to apply corresponding method of optimization, these equations should be added s equations for which constant parameters c_α are sufficient.

In this way, the motion of the given mechanical system can be described by means of the following 2n+s differential equations of the first order in the usual form (differential equations of the controlled motion):

$$\dot{q}^i = y^i,$$

$$\dot{y}^i = -\Gamma^i_{jk}y^j y^k + a^{ij}\left(-\frac{\partial \Pi}{\partial q^j} + Q^N_j\right), \qquad (3)$$

$$\dot{c}_\alpha = 0, \qquad i,k,j = 1,2,\ldots,n., \ \alpha=1, 2, \ldots, s,$$

where for the state values generalized coordinates q^i, generalized speeds y^i and constant parameters c_α are taken.

Generally, controls are limited. Limitations are the result of some subjectively imposed conditions, or of actual physical capabilities of the control system. The case when limitations have the form of equations will be discussed:

$$\Phi_\gamma(q^j, y^k, u_\beta, c_\alpha) = 0,$$

$$\gamma = 1,2,\ldots,p \quad j,k = 1,2,\ldots,n \quad \beta = 1,2,\ldots,m \quad \alpha = 1,2,\ldots s \qquad (4)$$

The task of optimal control by means of motion of the observed mechanical system with parameters consisits in determining constant parameters c_α and controls u_β from the group of permissive controls given in the form

[**] *Faculty of Traffic and Transport Engineering, University of Belgrade, Yugoslavia*

[*] *Faculty of Mechanical Engineering, University of Belgrade, Yugoslavia*

of relations (4) whose effect on the mechanical system will make the system, expressed in the form of differential equations system (3), from the initial state given on the basis of multiplicity:

$$\varphi_{\delta}^{0}\left[q^{j}(t_0), y^{k}(t_0), c_{\alpha}\right] = 0,$$

$$\delta = 1,2,\ldots,n_0 \leq 2n+s, \tag{5}$$

$$j,k = 1,2,\ldots,n,$$

$$\alpha = 1,2,\ldots,s,$$

reach the final state given on the basis of multiplicity:

$$\varphi_{l}^{1}\left[q^{j}(t_1), y^{k}(t_1), c_{\alpha}\right] = 0,$$

$$l = 1,2,\ldots,n_1 \leq 2n+s, \tag{6}$$

$$j,k = 1,2,\ldots,n,$$

$$\alpha = 1,2,\ldots,s,$$

on condition of minimum function:

$$I = \int_{t_0}^{t_1} f^{0}\left(q^{j}, y^{k}, u_{\beta}, c_{\alpha}\right) dt,$$

$$j,k = 1,2,\ldots,n, \tag{7}$$

$$\beta = 1,2,\ldots,m,$$

$$\alpha = 1,2,\ldots,s.$$

Function I represents optimality criterion.

2. SOLUTION TO THE PROBLEM BY MEANS OF THE MAXIMUM PRINCIPLE

Solution to the given assignment can be achieved by application of Pontryagin's principle of the maximum. For that purpose, on the basis of the equation system (3) Pontryagin's function is being introduced in the following form:

$$H\left(\lambda_0, \lambda_i, \nu_{\delta}, q^{j}, y^{k}, c_{\alpha}, u_{\beta}\right) = \lambda_0 f^{0} +$$
$$+ \lambda_i y^{i} + \nu_{\delta}\left[-\Gamma_{jk}^{\delta} y^{j} y^{k} + a^{\delta j}\left(-\frac{\partial \Pi}{\partial q^{j}} + Q_{j}^{N}\right)\right],$$

$$i, j, k, \delta = 1,2,\ldots,n,$$

$$\alpha = 1,2,\ldots,s,$$

$$\beta = 1,2,\ldots,m, \tag{8}$$

where $\lambda_0, \lambda_i i \nu_{\delta}$ are coordinates of the coupled vector.

On the ground of the expression (8) coupled equations can be formulated:

$$\dot{\lambda}_i = -\frac{\partial H}{\partial q^{i}} + \mu^{\gamma}\frac{\partial \Phi_{\gamma}}{\partial q^{i}},$$

$$\dot{\nu}_{\delta} = -\frac{\partial H}{\partial y^{\delta}} + \mu^{\gamma}\frac{\partial \Phi_{\gamma}}{\partial y^{\delta}}, \tag{9}$$

$$\dot{\eta}^{\alpha} = -\frac{\partial H}{\partial c_{\alpha}} + \mu^{\gamma}\frac{\partial \Phi_{\gamma}}{\partial c_{\alpha}},$$

$$i,\delta = 1,2,\ldots,n,$$

$$\alpha = 1,2,\ldots,s,$$

$$\gamma = 1,2,\ldots,p$$

where μ^{γ} are multipliers, and, on the basis of Pontryagin's theorem the expression $\lambda_0 = -1$ can be accepted.

Employing Pontryagin's principle of the maximum, i.e. of the following condition:

$$\frac{\partial H}{\partial u_{\beta}} = \mu^{\gamma}\frac{\partial \Phi_{\gamma}}{\partial u_{\beta}}. \tag{10}$$

$$\gamma = 1,2,\ldots,p,$$

$$\beta = 1,2,\ldots,m,$$

and the expression (4), and will eliminating the multiplier μ^{γ}, control can be obtained in the form as follows:

$$u_{\beta} = u_{\beta}\left(\lambda_i, \nu_{\delta}, q^{j}, y^{k}, c_{\alpha}\right), \tag{11}$$

$$i, j, k, \delta = 1,2,\ldots,n,$$

$$\alpha = 1,2,\ldots,s,$$

$$\beta = 1,2,\ldots,m.$$

When, in this way, obtained control is changed into differential equations of motion (3) and into the coupled system (9), the system of 4n+2s differential equations of the first order will be obtained in the usual form. For determination of the solution in the final form, it is necessary to have 4n+2s limitation conditions in case time t_1 is given. If a smaller number of limitation conditions is given, then transversality conditions are used [1]. On this basis, some coupled variables are equal to zero if in the initial and final conditions corresponding phase variables are not given. Solving the two-point limitation problem, optimum trajectories are obtained in the form:

$$q^j = q^j(t), \qquad y^k = y^k(t), \; c_\alpha = const,$$

$$\lambda_i = \lambda_i(t), \qquad \nu_\delta = \nu_\delta(t), \; \eta^\alpha = \eta^\alpha(t) \qquad (12)$$

$$i, j, k, \delta = 1,2,\ldots,n,$$

$$\alpha = 1,2,\ldots,s.$$

Substituting the solution (12) into the expression (11), programmed controls are obtained:

$$u_\beta = u_\beta(t), \qquad \beta = 1,2,\ldots,m. \qquad (13)$$

Such solutions represent extreme solutions, and if a greater number of them appears, the optimum ones should be sought.

3. APPLICATION OF SYMBOLIC PROGRAMMING AND OF MATHEMATICA INTERPRETER

Solution of the given assignment with the above presented algorithm requires a great number of routine procedures which demand lengthy calculation. Besides, mistakes typical of "manual" procedure can occur. Fortunately, nowadays, technique of symbolic programming makes possible production of programmes for symbolic performance of complex mathematical operations.

In this paper symbolic interpreter MATHEMATICA will be employed with reference to similar programme areas, such as interactive numerical systems (MATHCAD, MATLAB), or interactive algebraic systems (MACSYMA, MAPLE, REDUCE).

Differing from the classical programmes, which can be comprehended as a series of instructions on the basis of which out of one group of numbers the other group emerges, MATHEMATICA can be considered as a group of rules used as a basis for transforming expressions and formulae from one form into another. Nevertheless, MATHEMATICA is relatively slow and almost useless for intensive numerical calculations, so it is useful to include output results of the interpreter into the existing FORTRAN programmes. For this purpose there are commands FORTRAN Form [...], which transform corresponding expressions into the form suitable for FORTRAN syntax.

3.1. Programme for obtaining equations of two-point limitation problem with parameters and control limitations

```
aKont = Inverse[aKov];

krist[j_, k_, d_] := Sum[

        aKont[[d, l]] (

                D[aKov[[k, l]], q[j]] +

                D[aKov[[l, j]], q[k]] -

                D[aKov[[j, k]], q[l]]

        ) / 2,

        {l, 1, n}

];

H = Module[

        {sum1, sum2, sum3},

        sum1 = Sum[lam[i] y[i], {i, n}];

        sum2 = Sum[- ni[d] krist[j, k, d] y[j] y[k], {d, n},

                {j, n}, {k, n}];

        sum3 = Sum[ni[d] aKont[[d, j]] (-D[pi, q[j]] + Qgen[[j]]),

                {d, n}, {j, n}];

        Return[-f + sum1 + sum2 + sum3];

];

resenje = Module[

        {sist1, sist2, prom1, prom2, sist, prom},

        sist1 = Table[D[H, u[b]] - Sum[mi[g] D[fi[g], u[b]], {g,p}] == 0,

                {b,m}];

        sist2 = Table[fi[g] == 0, {g, p}];

        prom1 = Table[mi[g], {g, p}];

        prom2 = Table[u[b], {b, m}];

        sist = Join[sist1, sist2];

        prom = Join[prom1, prom2];


        Return[Solve[sist, prom]];

];

res = resenje[[1]];

qd[j_] := D[H, lam[j]] /. res;

yd[k_] := D[H, ni[k]] /. res;

cd[a_] := 0 /. res;
```

```
lamd[i_] := -D[H, q[i]] + Sum[mi[g] D[fi[g], q[i]], {g, p}] /. res;

nid[d_] := -D[H, y[d]] + Sum[mi[g] D[fi[g], y[d]], {g, p}] /. res;

etad[a_] := -D[H, c[a]] + Sum[mi[g] D[fi[g], c[a]], {g, p}] /. res;

pravilo = {q[i_] -> x[i], y[k_] -> x[n + k], c[a_] -> x[2n+a],

        lam[i_] -> x[2n+s+i], ni[d_] -> x[3n+s+d], eta[a_]->x[4n+s+a]};

xprime[i_] := (qd[i] /. pravilo) /; (i <= n);

xprime[i_] := (yd[i - n] /. pravilo) /; (i >= n + 1 && i <= 2n);

xprime[i_] := (cd[i - 2n] /. pravilo) /; (i >= 2n + 1 && i <= 2n + s);

xprime[i_] := (lamd[i - 2n - s] /. pravilo) /; (i >= 2n + s + 1 && i <= 3n + s);

xprime[i_] := (nid[i - 3n - s] /. pravilo) /; (i >= 3n + s + 1 && i <= 4n + s);

xprime[i_] := (etad[i - 4n - s] /. pravilo) /; (i >= 4n + s + 1);

pd[i_, j_] := D[xprime[i], x[j]];
```

The given programme contains determination of Jacobians, necessary for numerical way of solving. Finite differences method gives very good results in solving the problems of this kind [1].

4. EXAMPLE

Figure 1. shows mechanical model of a lift.

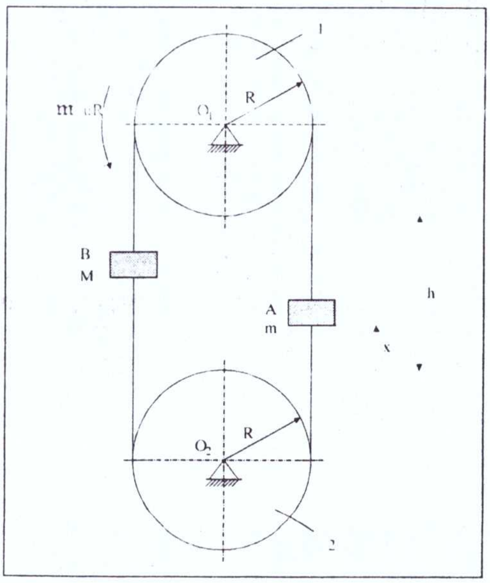

Figure 1.

It is necessary to determine coupling **m** in the function of time which is to act onto disc 1, and on mass M of the counterweight B in order to make load A, of mass m=200 kg move from the initial state:

$$t_0 = 0, \qquad x(0) = 0, \qquad \dot{x}(0) = 0, \tag{14}$$

and reach the final state:

$$t_1 = 10[s], \quad x(10) = h = 20[m], \quad \dot{x}(10) = 0, \tag{15}$$

on condition of the function minimum:

$$I = \frac{1}{2}\int_0^{t_1} u^2 dt, \tag{16}$$

where the control u is given as:

$$u = \frac{\mathrm{m}}{R}. \tag{17}$$

Ropes mass and masses of discs are of minor value. Friction moment in bearings O_1 and O_2 can be neglected. It will be considered that there is no slippage between the rope and discs.

Function (16) is used when minimization of energy that converts into heat within electric motor coils is needed, and the explanation is given in [1].

Given mechanical system has one degree of free movement, and for generalized coordinate motion x of the load A in relation to the initial state is taken.

Kinetic energy of the system is:

$$T = \frac{1}{2}(M + m)\dot{x}^2, \tag{18}$$

while potential energy is given by the expression:

$$\Pi = mgx + Mg(h - x), \tag{19}$$

Differential equation of the system motion (1) has the following form:

$$\ddot{x} = \frac{(M - m)g + u}{M + m}, \tag{20}$$

and differential equations of the controlled motion (3) are of the following form:

$$\dot{q}^1 = y^1,$$

$$\dot{y}^1 = \frac{(c_1 - m)g + u}{c_1 + m}, \tag{21}$$

$$\dot{c}_1 = 0.$$

Now, Pantryagin's function can be formulated (See (8)):

$$H=-\frac{1}{2}u^2+\lambda_1 y^1+\nu_1\frac{(c_1-m)g+u}{c_1+m}, \qquad (22)$$

and on its basis the coupled equation system can be expressed (See (9)):

$$\dot{\lambda}_1=0,$$

$$\dot{\nu}_1=-\lambda_1, \qquad (23)$$

$$\dot{\eta}^1=\nu_1\frac{u-2mg}{(c_1+m)^2}.$$

Applying Pontryagin's principle of the maximum, i.e. of conditions (10), in case of control out of the open group, control in the form (11) is obtained:

$$u=\frac{\nu_1}{m+c_1}. \qquad (24)$$

With substitution of the expression (24) into equations (21) and (23) the system of six differential equations of the first order is obtained in the usual form:

$$\dot{q}^1=y^1,$$

$$\dot{y}^1=\frac{\nu_1}{(m+c_1)^2}+\frac{(c_1-m)g}{m+c_1},$$

$$\dot{c}_1=0,$$

$$\dot{\lambda}_1=0, \qquad (25)$$

$$\dot{\nu}_1=-\lambda_1,$$

$$\dot{\eta}^1=\frac{\nu_1^{\,2}}{(m+c_1)^3}-\frac{2mg\nu_1}{(m+c_1)^2}.$$

Initial conditions (14) and final conditions (15) can be expressed in the following form:

$$q^1(0)=0, \qquad y^1(0)=0,$$

$$q^1(10)=20[m], \qquad y^1(10)=0, \qquad (26)$$

while, on the basis of transversality (See [1]):

$$\eta^1(0)=0, \qquad \eta^1(10)=0. \qquad (27)$$

Using limitation conditions (26) and (27) solutions to the differential equations system (25) can be obtained in the form:

$$q^1=q^1(t),\ y^1=y^1(t),\ c_1=const,\ \lambda_1=const,$$

$$\nu_1=\nu_1(t),\ \eta^1=\eta^1(t). \qquad (28)$$

With substitution of the previous solutions in (24) programmed optimal control is obtained $u=u(t)$.

The solution is obtained (numerically) applying the method of finite differences. Diagrams of phase variables are shown on Fig. 2, and diagram of control is shown on Fig. 3.

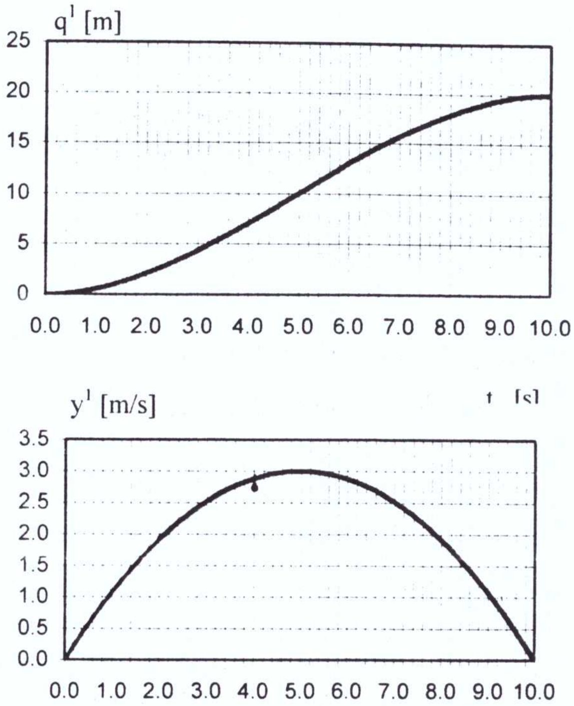

Figure 2.

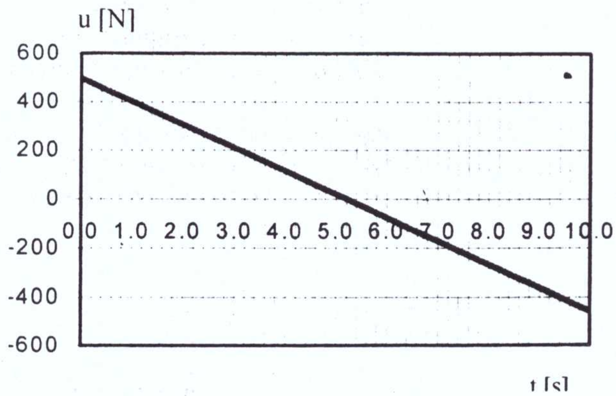

Figure 3.

REFERENCES:

[1] Marković Z., *Rešavanje problema optimalnog upravljanja i parametarske optimizacije mehaničkih sistema metodom konačnih razlika*, magistarska teza, Mašinski fakultet Beograd, 1997.

[2] Bryson A. E., Ho Y. C., *Applied Optimal Control, Ginn and Company*, Waltham, 1969.

[3] Demidovich B. P., Maron I. A., *Computational Mathematics*, Mir Publishers, Moscow, 1981.

[4] Pereyra V., PASVA 3, *An Adaptive Finite Difference Fortran Program for First Order Non - Linear Ordinary Boundary Problems*, Lecture Notes in Computer Science, Springer Verlag, Berlin, 1978.

[5] Sage A.P., White C.C., *Optimum Systems Control,* Prentice - Hall, Englewood, 1977.

[6] Vuković J., Obradović A., *Determination of Constant Parameters of Optimally controlled Systems*, Transactions, Belgrade, No 1, 1996.

[7] William H. Press, Brian P. Flannery, Saul A. Teukolsky, Wiliam T. Vetterling, *Numerical Recipes in C*, Cambridge Universiti Pres, 1988.

[8] Барков В.В., Кочетков Ю.А., *Краевая задача оптимального управления нелинейными детер минированными системами*, Известия РАН, Теория и системы управления, No 6, 1995.

[9] Григорьев К. Г., Заплетин М. П., *Численное решение kраевых задач принципа максимума в оптимизационных задачах динамики kосмичес kого полета*, Известия РАН ТК, No 1, 1993.

[10] Григорьев К.Г., Заплетин М.П., *Численное решение kраевых задач принципа максимума в оптимизационных задачах динамики kосмичес kого полета*, Известия РАН ТК, No 1, 1993.

[11] Алексеев В.М., Тихомиров В.М., Фомин С. В., *Оптимальное управление*, Hayka, Москва, 1979.

[12] Понтрягин Л.С., Болтянский В.Г., Гамkрелид зе Р.В., Мищенко Е.Ф., *Математическая тео рия оптимальных процессов*, Hayka, Москва, 1983.

Numeričko rešavanje problema optimalnog upravljanja i parametarske optimizacije mehaničkih sistema

Z. Marković, A. Obradović, J. Vuković

U radu je dat algoritam za izračunavanje programskih optimalnih upravljanja i konstantnih parametara mehaničkih sistema, kada su nametnuta ograničenja upravljanja i faznih promenljivih tipa jednakosti. Postupak je zasnovan na principu maksimuma. Uslovi principa maksimuma dopunjeni su odgovarajućim uslovima transverzalnosti. Dvotačkasti granični problem principa maksimuma rešen je numerički, metodom konačnih razlika. Primenom simboličkog programiranja izbegnute su moguće greške svojstvene pri tzv. "ručnom" manipulisanju simboličkim izrazima pri dobijanju diferencijalnih jednačina i odgovarajućih Jakobijana. Postupak je ilustrovan primerom.

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Numerical Solution to Optimal Control Problems and Parameter Optimization of Mechanical Systems

Z. Marković, A. Obradović, J. Vuković

In this paper algorithm for calculation of programmed optimal controls and mechanical systems constant parameters is given, when limitation to the control and phase variables of the equation type are imposed. The procedure is based on the maximum principle. Conditions of the maximum principle are supplemented with corresponding conditions of transversality. Two-point limitation problem of the maximum principle is numerically solved using the method of finite differences. With symbolic programming, possible mistakes, typical for so called "manual" manipulation of symbolic expressions in obtaining differential equations and corresponding Jacobians, were avoided. The procedure has been illustrated with an example.

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