BRACHISTOCHRONIC MOTION OF A VARIABLE MASS SYSTEM

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Abstract. A mechanical system consisting of rigid bodies and material particles, of which some particles are with variable masses, is considered. Laws of variation of the masses of the points and relative velocity of particles separating from the points are well-known. The system is moving in an arbitrary field of known potential and nonpotential forces. Applying Pontryagin's Maximum Principle and singular optimal control theory, brachistochronic motion is determined. A two-point boundary value problem, due to nonlinearity of equations in a general case, is needed to be solved using some of the numerical procedures. Here the Shooting method is used, where the missing boundary conditions are chosen so as to be the physical variables (velocity and mass). The field where they are found can be approximately estimated, which is not the case with the conjugate vector coordinates being of purely mathematical nature. The paper also presents the manner of brachistrochronic motion realization without the action of active control forces. It is realized by subsequent imposition to the system a corresponding number of independent ideal holonomic mechanical constraints. The constraints must be in accordance with the previously determined brachistochronic motion of the system. The method is illustrated by an example of determining the brachistochronic motion of the system with three degrees of freedom and method of its realization. The system consists of one rigid body to which two points of variable masses are attached, where the system is moving in a vertical plane. Brachistochronic motion is realized by the help of two ideal holonomic constraints.

1. Introduction

The problem of a brachistochronic motion of mechanical systems is a very topical area of research as evidenced from literature cited. Research is inspired not only by the expansion of existing fundamental knowledge in this area, but also by various engineering applications (see e.g., [1-7]). Thus in [8-16] the brachistochronic motion of a particle in the presence of resistance forces (forces of dry friction, viscous friction) is analyzed, while in [6,17,18] the brachistochronic motion of a particle on a surface is considered. In [18] it was shown that results from [8,10,13] represent special cases of the brachistochronic motion of a particle on a surface. Note that in [6] the problem of optimization of a bobsled travelling on a path was solved as the problem of a brachistochronic motion of a particle on a surface, whereas [7] considers the brachistochrone problem for a steerable particle moving on a 1D curved surface
with application to ski racing. The next important group of references comprises the papers that consider the problem of brachistochronic motion of a rigid body [1-3,5,19] and system of rigid bodies [21-23]. Furthermore, in [24-27] the brachistochronic motion of mechanical systems with nonholonomic constrains is analyzed. Also, a certain number of references

can be singled out [14,28-31], where the solution of the classical brachistochrone problem (cycloid) was used with the aim of testing various numerical methods in solving nonlinear engineering optimization problems. References [32,33] consider the problem of brachistochronic motion of a variable mass particle.
This paper considers the mechanical systems that consist of constant-mass rigid bodies and variable-mass particles. It is started from the assumption that such mechanical systems are moving in the arbitrary field of known potential and nonpotential forces. Pontryagin's Maximum Principle [34] and singular optimal control theory [35] is applied in solving the brachistrochrone problem. Considerations in this paper represent a continuation of research commenced in paper [27].

2. Problem statement

The motion of mechanical system with n-degrees of freedom within which there are ℓ variable-mass particles is considered. The system configuration is determined by generalized coordinates $\overline{q} = (q^1, q^2, \ldots, q^n)$. Laws of variation of the masses are well-known:

$$m_\rho = m_\rho(t), \quad \rho = 1, \ldots, \ell \tag{1}$$

as well as relative velocity of particles separating from the points

$$\vec{v}_\rho^{\,rel} = \vec{v}_\rho^{\,rel}(\overline{q}, \dot{\overline{q}}, t), \qquad \rho = 1, \ldots, \ell. \tag{2}$$

Since the system motion is under the imposition of holonomic scleronomic mechanical constraints, kinetic energy has the form

$$T = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j, \qquad i, j = 1, \ldots, n \tag{3}$$

where covariant coordinates of metric tensor, taking into account (1), are

$$a_{ij} = a_{ij}(\overline{q}, t), \qquad i, j = 1, \ldots, n. \tag{4}$$

Let the system move in the field of known potential forces with potential energy

$$\Pi = \Pi(\overline{q}, t) \tag{5}$$

and let the system be also acted upon by arbitrary known nonpotential forces, whose generalized forces are

$$Q_i^w = Q_i^w(\overline{q}, \dot{\overline{q}}, t), \qquad i = 1, \ldots, n. \tag{6}$$

Differential equations of motion of such variable-mass system in the form of Lagrange's equations of the second kind [36] have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = -\frac{\partial \Pi}{\partial q^i} + Q_i^w + Q_i^{\mathrm{var}} + Q_i^c \tag{7}$$

where generalized forces Q_i^{var} have the following form

$$Q_i^{\mathrm{var}}(\bar{q},\dot{\bar{q}},t)=\sum_{\rho=1}^{\ell}\dot{m}_\rho(\vec{v}_\rho+\vec{v}_\rho^{\,rel})\frac{\partial\vec{r}_\rho}{\partial q^i}, \tag{8}$$

while Q_i^c represent generalized control forces. Their determination represents an essential part of solving the problem of brachistochronic motion of the mechanical system. They can be generalized active forces and/or reactions of constraints, depending on the manner of brachistochronic motion realization. In accordance with the original postulates of brachistochronic motion [20] their power equals zero

$$Q_i^c\dot{q}^i=0,\qquad i=1,\ldots,n\,. \tag{9}$$

Thus based on (3), (7) and (9), there exists linear dependence of the second derivatives of generalized coordinates

$$a_{ij}\ddot{q}^j\ddot{q}^i=\left(-\dot{a}_{ij}\dot{q}^j+\frac{\partial T}{\partial q^i}-\frac{\partial\Pi}{\partial q^i}+Q_i^w+Q_i^{\mathrm{var}}\right)\dot{q}^i,\qquad i,j=1,\ldots,n. \tag{10}$$

Let the initial values of generalized coordinates and total mechanical energy of the system be specified

$$t_0=0,\quad \bar{q}(t_0)=\bar{q}_0,\quad T(\bar{q}_0,\dot{\bar{q}}_0,t_0)+\Pi(\bar{q}_0,t_0)=E_0 \tag{11}$$

as well as terminal values of generalized coordinates at unknown instant of time t_1

$$\bar{q}(t_1)=\bar{q}_1\,. \tag{12}$$

Solving the problem of brachistochronic motion of a variable-mass mechanical system, whose differential equations are (7), consists in determining the control forces $Q_i^c=Q_i^c(t)$ and the system motions corresponding to them, so that the system transfers for the shortest time from the state described by (11) into the state described by (12).

2. Brachistochronic motion as a problem of optimal control

Linear constraint (10) allows for another derivative of generalized coordinate to be expressed via the others. Let it be, without limiting the generality

$$\ddot{q}^n=\Phi+\Phi_s\ddot{q}^s\qquad s=1,\ldots,n-1 \tag{13}$$

where:

$$\Phi(\bar{q},\dot{\bar{q}},t)=\frac{\left(-\dot{a}_{ij}\dot{q}^j+\dfrac{\partial T}{\partial q^i}-\dfrac{\partial\Pi}{\partial q^i}+Q_i^w+Q_i^{\mathrm{var}}\right)\dot{q}^i}{a_{in}\dot{q}^i},$$

$$\Phi_s(\bar{q},\dot{\bar{q}},t)=\frac{-a_{is}\dot{q}^s}{a_{in}\dot{q}^i},\quad i,j=1,\ldots,n;\ s=1,\ldots,n-1. \tag{14}$$

Introducing the control

$$u^s = \ddot{q}^s \qquad s = 1,\ldots,n-1 \tag{15}$$

differential equations of the first kind in the problem of optimal control can be, incorporating the rheonomic coordinate, written in the form

$$\dot{q}^i = y^i,\ \dot{q}^{n+1} = 1,\ \dot{y}^s = u^s,\ \dot{y}^n = \Phi + \Phi_s u^s. \tag{16}$$

Taking into account the form of functional in a time minimization problem

$$J = \int_{t_0}^{t_1} dt, \tag{17}$$

In solving the problem by the help of Pontrryagin's Maximum Principle [34], it is necessary to form Pontryagin's function

$$H = \lambda_0 + \lambda_i \dot{y}^i + \kappa_s u^s + \kappa_n(\Phi + \Phi_s u^s) + \lambda_{n+1}, \quad i = 1,\ldots,n,\ s = 1,\ldots,n-1 \tag{18}$$

where λ_0, λ_i, λ_{n+1}, and κ_i are coordinates of the conjugate vector. A costate system of differential equations corresponds to them

$$\begin{aligned}
\dot{\lambda}_i &= -\frac{\partial H}{\partial q^i} = -\kappa_n\left(\frac{\partial \Phi}{\partial q^i} + \frac{\partial \Phi_s}{\partial q^i}u^s\right)\\
\dot{\lambda}_{n+1} &= -\frac{\partial H}{\partial q^{n+1}} = -\kappa_n\left(\frac{\partial \Phi}{\partial q^{n+1}} + \frac{\partial \Phi_s}{\partial q^{n+1}}u^s\right) \qquad .\\
\dot{\nu}_i &= -\frac{\partial H}{\partial y^i} = -\lambda_i - \kappa_n\left(\frac{\partial \Phi}{\partial y^i} + \frac{\partial \Phi_s}{\partial y^i}u^s\right), \quad i = 1,\ldots,n,\ s = 1,\ldots,n-1
\end{aligned} \tag{19}$$

Pontryagin's function (18) depends linearly on the control

$$H = H_0 + H_s u^s \quad s = 1,\ldots,n-1 \tag{20}$$

In the optimal control theory such case is referred to as singular [35] because the corresponding condition of a maximum principle

$$\frac{\partial H}{\partial u^s} = H_s = 0 \quad s = 1,\ldots,n-1 \tag{21}$$

does not allow for determining the optimal controls. Instead, one obtains a constraint between parts of the conjugate vector coordinates

$$\kappa_s = -\kappa_n \Phi_s \qquad s = 1,\ldots,n-1 \tag{22}$$

In order to determine optimal controls, it is necessary to further differentiate the relation (21) with respect to time in accordance with (16) and (19). Applying the formalism of Poisson brackets [36]

$$\dot{H}_s = \{H, H_s\} = \{H_s, H_0\} + \{H_s, H_z\}u^z = 0, \qquad s, z = 1,\ldots,n-1 \tag{23}$$

and taking into account the fact that for multidimensional singular controls [35]

$$\{H_s, H_z\} = 0, \quad s, z = 1, \ldots, n-1 \tag{24}$$

it is obtained that

$$\{H_s, H_0\} = 0, \quad s = 1, \ldots, n-1 \tag{25}$$

From (25), taking into account (22), another constraint between the coordinates of conjugate vector can be established

$$\lambda_s = \lambda_s(\bar{q}, \bar{y}, t, \lambda_n, \kappa_n) \quad s = 1, \ldots, n-1 \tag{26}$$

Further differentiation yields:

$$\{\{H_s, H_0\}, H_0\} + \{\{H_s, H_0\}, H_z\} u^z = 0, \quad s, z = 1, \ldots, n-1 \tag{27}$$

Limiting to the singular controls of the first order [35], the linear system of equations (27) using the relations (22) and (26) yields optimal controls

$$u^s = u^s(\bar{q}, \bar{y}, t, \lambda_n, \kappa_n), \quad s = 1, \ldots, n-1 \tag{28}$$

Substituting (28) in (16) and (19) one obtains the system of (2n+2) differential equations of the first kind in normal form

$$\begin{aligned}
\dot{q}^i &= \dot{q}^i(\bar{q}, \bar{y}, t, \lambda_n, \kappa_n), \\
\dot{y}^i &= \dot{y}^i(\bar{q}, \bar{y}, t, \lambda_n, \kappa_n), \\
\dot{\lambda}_n &= \dot{\lambda}_n(\bar{q}, \bar{y}, t, \lambda_n, \kappa_n), \\
\dot{\kappa}_n &= \dot{\kappa}_n(\bar{q}, \bar{y}, t, \lambda_n, \kappa_n), \quad i = 1, \ldots, n
\end{aligned} \tag{29}$$

where differential equations, whose solutions are (22) and (26), were eliminated from (19). In a general case, due to nonlinearity (29), a two-point boundary value problem should be solved by applying some of the numerical methods. If the Shooting method is used [37], it is necessary to adjust the choice of the missing boundary conditions such that one can approximately estimate their field. In this regard, it should be avoided, whenever it is possible, having any of the coordinates of conjugate vector among them, because they are of purely mathematical character and as such difficult to estimate the field. Therefore, it is suitable here to perform backward numerical integration in the interval $[t_0, t_1]$. At terminal point the maximum principle can be utilized for the case of unspecified interval $[t_0, t_1]$

$$H(t_1) = 0 \tag{30}$$

Since final velocities $y^i(t_1)$ are not specified, nor is the rheonomic coordinate $q^{n+1}(t_1)$, it follows from the transversality conditions that:

$$\kappa_i(t_1) = 0, \; i = 1, \ldots, n, \quad \lambda_{n+1}(t_1) = 0 \tag{31}$$

Taking into account that $\lambda_0 = -1$ (according to the maximum principle $\lambda_0 = const \le 0$) as well as the relations (12),(18),(26),(30) and (31), it is possible to establish in the analytical form the dependence

$$\lambda_n(t_1) = \lambda_n(\bar{y}_1, t_1) \tag{32}$$

which, along with the fact that

$$\kappa_n(t_1) = 0 \tag{33}$$

completely excludes the necessity to estimate the fields for $\lambda_n(t_1)$ and $\kappa_n(t_1)$ in backward shooting procedure.

The backward shooting procedure consists of choosing n+1 values of generalized velocities $y^i(t_1)$ and duration t_1 of the time interval $[t_0, t_1]$, so that n+1 values (11) of generalized coordinates and mechanical energy are shot.

There remains the discussion on transversality conditions at the initial point:

$$\lambda_i(t_0)\delta q^i(t_0) + \kappa_i(t_0)\delta y^i(t_0) + \lambda_{n+1}(t_0)\delta q^{n+1}(t_0) = 0 \quad i = 1,\ldots,n \ . \tag{34}$$

Based on specified values (11) it follows

$$\delta q^i(t_0) = 0 \ , \qquad \delta q^{n+1}(t_0) = 0, \quad a_{ij}(t_0)y^i(t_0)\delta y^i(t_0) = 0, \quad j,i = 1,\ldots,n \ . \tag{35}$$

Substituting (14) in (22), it is obtained

$$\kappa_i(t_0)\delta y^i(t_0) = \kappa_n(t_0)a_{ij}(t_0)y^j(t_0)\delta y^i(t_0) \quad i,j = 1,\ldots,n \ . \tag{36}$$

Directly substituting (35) and (36) in (34), it is evident that transversality conditions at the initial point are satisfied. Numerical solving of the system (29) yields

$$\overline{q} = \overline{q}(t),\ \dot{\overline{q}} = \dot{\overline{q}}(t),\ \lambda_n = \lambda_n(t),\ \kappa_n = \kappa_n(t) \tag{37}$$

and based on (13), (28) it is also obtained

$$\ddot{\overline{q}} = \ddot{\overline{q}}(t) \tag{38}$$

which enables too final determination of the control forces (7)

$$Q_i^c = Q_i^c(t) \tag{39}$$

Control forces (39) can be realized in various ways, combining active forces and/or reactions of constraints. The most approximate to the original brachistochrone problem is realization of motion by subsequent imposition to the system a corresponding number of independent ideal stationary constraints, without the action of active forces. The constraints must be in accordance with the brachistochronic motion (37).

Let the ideal holonomic stationary independent constraints, in accordance with (37), be imposed to the system

$$\varphi^s(\overline{q}) = 0,\ rank\left[\frac{\partial \varphi^s}{\partial q^i}\right] = n-1,\ s = 1,\ldots,n-1 \tag{40}$$

In that case, generalized control forces read

$$Q_i^c = \mu_s \frac{\partial \varphi_s}{\partial q^i}, \quad i = 1,\ldots,n; \quad s = 1,\ldots,n-1 \tag{41}$$

where from, if necessary, based on (37), (39) and (40), multipliers of constraints can be also determined

$$\mu_s = \mu_s(t) \quad s = 1,\ldots,n-1 \tag{42}$$

More information on such manner of control can be found in []. The form of constraints (40) is most often chosen to be performed by the simplest construction. One of the manners for the case of motion control of mechanical systems, especially of a rigid body, is imposition of guides to the specified number of particles whose motion is determined by numerical relations.

3. Example

The rod AB of mass *m*, of length 2ℓ and radius of inertia $i_{Cz} = \ell$ moves in a vertical plane, where the Oy axis is directed upward (see Fig.1).

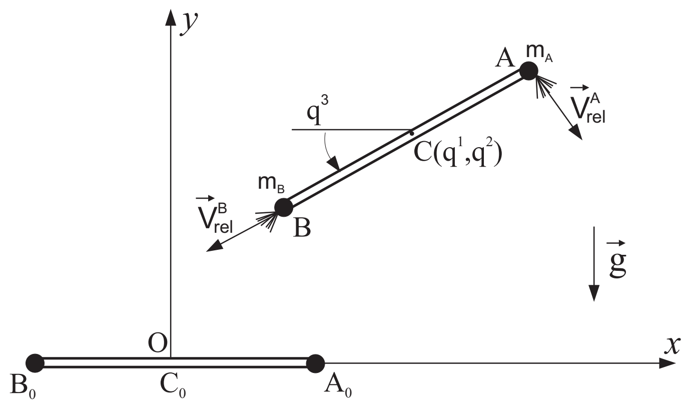

Figure 1. Variable-mass mechanical system

At both ends of the rod there are two variable-mass points, whose masses change according to the Law

$$m_A(t) = m_B(t) = m - kt \tag{43}$$

where $k = const > 0$. The particles are separating by relative velocities of constant intensities $(v = const > 0)$

$$v^B_{rel} = v^A_{rel} = v \tag{44}$$

It is needed to determine the brachistochronic motion of the system and present its realization without the action of active forces if at the initiation of motion (11) $(E_0 > 0)$ is specified:

$$t_0 = 0, \qquad q^1(t_0) = q^2(t_0) = q^3(t_0) = 0,$$

$$\frac{3m}{2}\left[\left(\dot{q}^1(t_0)\right)^2+\left(\dot{q}^2(t_0)\right)^2+\left(\dot{q}^3(t_0)\right)^2\right]=E_0\,, \tag{45}$$

while at the end of motion (12)

$$t_1=?,\quad q^1(t_1)=q^2(t_1)=\ell,\quad q^3(t_1)=\frac{\pi}{2} \tag{46}$$

Differential equations of motion (7) of this system are:

$$\begin{aligned}&(3m-kt)\ddot{q}^1=kv(\cos q^3-\sin q^3)+Q_1^c\\&(3m-kt)\ddot{q}^2=kv(\cos q^3+\sin q^3)-(3m-kt)g+Q_2^c\\&(3m-kt)\ell^2\ddot{q}^3=kv\,\ell+Q_3^c\end{aligned} \tag{47}$$

so that the relations (14) obtain the form

$$\begin{aligned}&\Phi=\frac{1}{\ell^2\dot{q}^3}\left[-g+\tfrac{kv}{3m-kt}\left(\cos q^3(\dot{q}^1+\dot{q}^2)+\sin q^3(\dot{q}^2-\dot{q}^1)+\ell\dot{q}^3\right)\right],\\&\Phi_1=-\frac{\dot{q}^1}{\ell^2\dot{q}^3},\quad \Phi_2=-\frac{\dot{q}^2}{\ell^2\dot{q}^3}\end{aligned} \tag{48}$$

The problem is solved for the following numerical values of the parameters:

$$\ell=1m,\quad m=1kg,\quad E_0=30\,J,\; v=1\frac{m}{s},\; k=1\frac{kg}{s}\,. \tag{49}$$

The missing values of boundary conditions are:

$$\begin{aligned}&\dot{q}^1(t_1)=1.14877\frac{m}{s}\quad,\dot{q}^1(t_2)=1.30177\frac{m}{s},\,.\\&\dot{q}^1(t_3)=2.94686\,s^{-1}\quad,t_1=0.532857\,s\end{aligned} \tag{50}$$

The trajectories of points A , B , and C are shown in Fig.2.

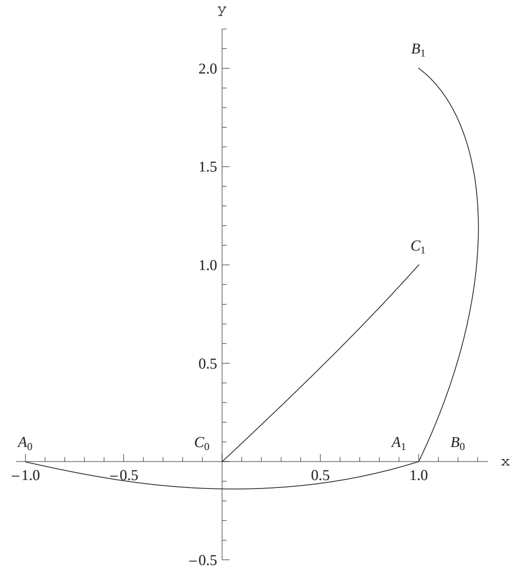

Figure 2: Trajectories of points A, B, and C

4. Conclusions

This paper is a continuation of research from [27] for the case of brachistochronic motion of a variable-mass system. Like in [27], the manner of motion control is presented without the action of active forces. The novelty in this paper is the numerical solving procedure for the two-point boundary value problem of maximum principle, based on shooting method, where costate variables were avoided as the missing boundary conditions. The number of missing boundary conditions is the least possible, such as $n-$generalized velocities and time t_1, which yields $n+1$ conditions. Their values can be approximately estimated.

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