The brachistochronic motion of a heavy ball rolling along an imperfect rough surface*

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ABSTRACT

The problem of brachistochronic motion of a heavy uniform ball rolling without slip along the upper outside surface of an imperfect rough stationary sphere, is solved. The control forces are located in the tangential plane, and their total power equals zero. In the first part of the paper the determination of the brachistochronic motion is solved as the problem of optimal control using Pontryagin's maximum principle. This solution corresponds to the motion of the heavy ball along a perfect rough sphere. The second part provides the case when the constraint between the sphere and the ball is imperfectly rough. Here, the problem of optimal control is formulated in such way that the tangential component of the reaction of constraint is taken for the control, with the restriction resulting from Coulomb's laws of sliding friction. The problem thus formulated belongs to the theory of singular optimal controls, and the solution that satisfies the Maximum principle consists of a singular part and a non-singular part.

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1. Introduction

The classical brachistochrone problem of the heavy point in a vertical plane has experienced application to various classes of mechanical systems over the past years. Among the contemporary works, the doctoral dissertation [1] should be noted, which proves the actuality of the problem even today. This recently defended dissertation reveals that the problems of the brachistochronic motions involve the areas not investigated yet. The dissertation provides a very good survey of the results which include several of our papers as well. In creating the task of optimal control, as we do in our papers, some of the reaction of constraints is taken for the control. The same has been done in [2]. The brachistochronic motion of the heavy point under the action of the sliding friction force was also the subject of the research in both [3] and [4], where bilaterally limited normal reaction of the constraint was studied.

In [5] both the classical brachistochrone problem and the unrestrained brachistochrone problem were formulated within the framework of the optimal control theory. Also, the extension to the cases of the brachistochrone problem for a rolling rigid body in a vertical plane, as well as the three-dimensional-minimum-time optimal problem for a disk rolling on the interior surface of a hemisphere was given.

Here, the research focuses on those brachistochrone problems of the nonholonomic mechanical systems where the reaction of the nonholonomic constraint is limited, and the paper is a continuation of the authors' earlier research studies.

In [6] the lateral reaction of the constraint ($|R_\eta| < N_b$) of the Chaplygin sleighis bilaterally limited (Fig. 1), and the brachistochronic motion is realized by subsequent imposition of an ideal holonomic constraint to the mass center C. This type of restriction was considered in a well-known work by Caratheodory [7].

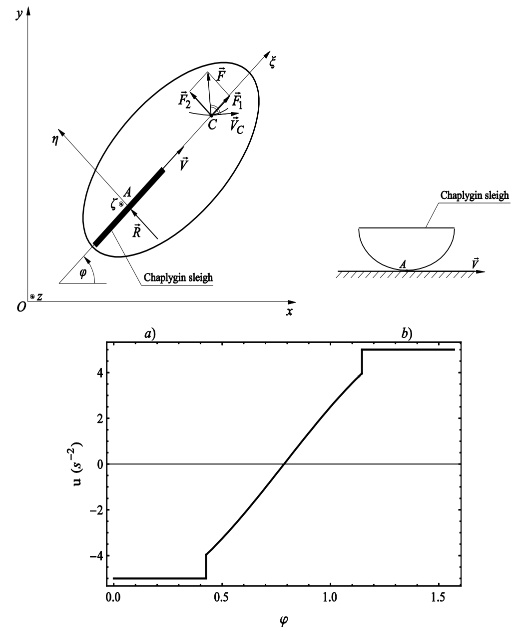


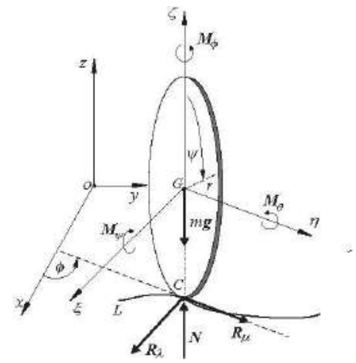
Fig.1 [6] Brachistochronic motion of the Chaplygin sleigh

The brachistochronic motion is realized in such way that on some parts the constraint reaction is on one of its limits $R_\eta = \pm N_b$ and a singular part is in the middle of motion, so that angular acceleration, depending on the angle, is given in Fig. 1c.

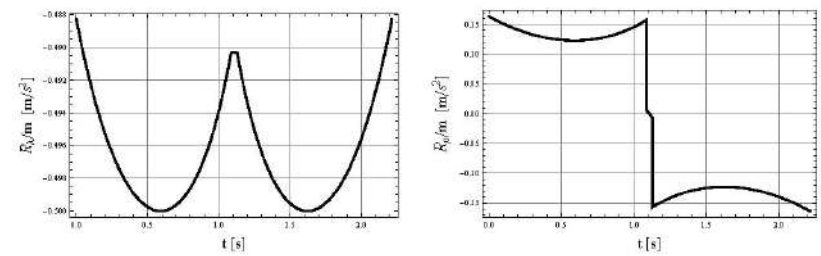
In the cases when the disks are rolling along horizontal surfaces Coulomb's laws of sliding friction limit the maximum possible horizontal components of the constraint reactions.

Paper [8] considers the brachistochronic motion of a vertical disk along a horizontal plane. Motion is controlled by three couples and the restrictions arise from the condition that slip will not occur at the contact point of the disk and the surface. Figure 2b) shows change in the horizontal components of the disk constraint reactions in the numerical example, where a non-singular part is in the middle of the interval of motion.

Previous research was extended in [10] to a more complex system of bodies, the simplified model of a vehicle [9], Fig. 3. Wheels slip is prevented based on the restrictions following from Coulomb's laws of friction. For real values of Coulomb's coefficient of friction, in this case too, one obtains non-singular parts of the brachistochronic motion, where a horizontal component of the reaction force is on its limit.

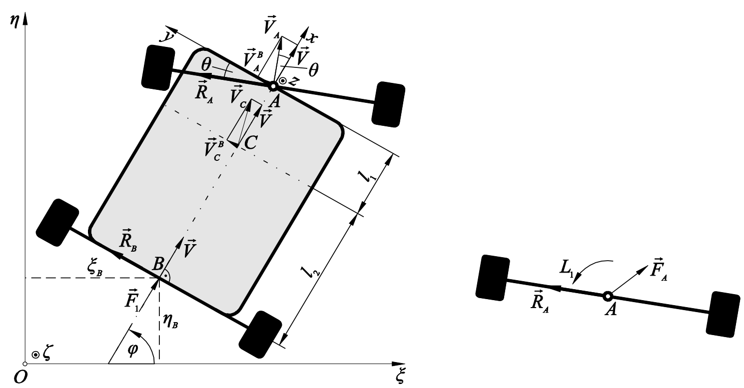


a)



b)

Fig. 2 [8] Brachistochronic motion of a vertical disk



$$\mu_2(t)N_2 = |R_A| < F_2^{fr} = \mu N_2, \quad \mu_1(t)N_1 = F_b = \sqrt{R_B^2 + F_1^2} < F_1^{fr} = \mu N_1$$

Fig. 3 [10] Brachistochronic motion of a simplified vehicle model [9]

The major goal of this paper is to determine the brachistochronic motion of the heavy ball rolling along a real rough sphere. For various real materials that the ball and the sphere can be made of their contact cannot be considered perfectly rough. In real rough contact the quotient of the intensity of horizontal and vertical component of the constraint reaction must be lower than the real value of Coulomb’s friction coefficient. It is necessary to apply the procedure similar to that for the rolling disk [8].

2. Problem formulation

Observe the motion of a heavy uniform ball rolling without slip along the upper outside surface of an imperfect rough stationary sphere (Fig. 4)

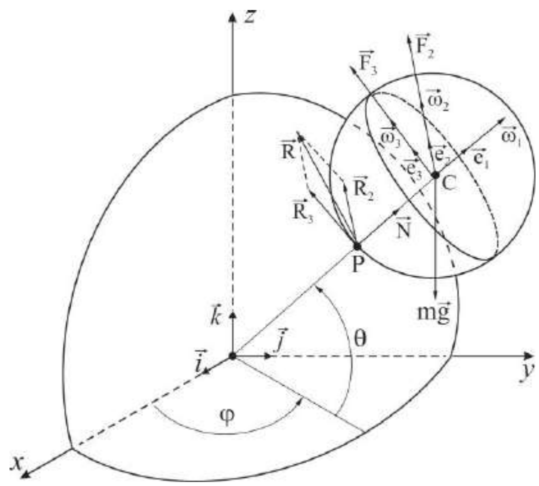


Fig.4 Heavy ball rolling without slip along the upper outside surface of a stationary sphere($\overline{OC} = L,\ \ \overline{PC} = r$)

The control forces F_2 and F_3 are located in the tangential plane of the other sphere, along which the center of the ball is moving, and their total power equals zero in the brachistochronic motion of the mechanical systems. Thus, during motion the total mechanical energy is maintained. The initial value of the mechanical energy is specified, and the initial and final position of the ball's center is defined by the spherical coordinate system angles φ and θ, the ball orientation (Euler angles) not being considered in this problem. Such mechanical system is nonholonomic, and dynamic equations in this paper are derived using the general theorems of mechanics.
The theorems on the motion of the center of mass and the change of the kinetic moment for the center of mass read:

$$m\vec{a}_C = \vec{F}_R^S, \qquad \dot{\vec{L}}_C = \vec{M}_C^S, \tag{1}$$

where the velocities and accelerations of the mass center in this coordinate system are:

$$\begin{aligned}&v_{C1} = 0, v_{C2} = L\dot{\varphi}\cos\theta\,, v_{C3} = L\dot{\theta}\\&a_{C1} = -L\left(\dot{\varphi}^2 cos^2\theta + \dot{\theta}^2\right), a_{C2} = L\left(\ddot{\varphi}\cos\theta - 2\dot{\varphi}\,\dot{\theta} sin\,\theta\right),\\&a_{C3} = L\left(\ddot{\theta} + \dot{\varphi}^2 \sin\theta\cos\theta\right),\end{aligned} \tag{2}$$

and dynamic quantities in expressions (1) are given by expressions:

$$\vec{L}_C = J\vec{\omega},\ \ J = \frac{2}{5}mr^2,$$
$$\vec{F}_R^S = m\vec{g} + \vec{N} + \vec{F}_2 + \vec{F}_3 + \vec{R}_2 + \vec{R}_3, \tag{3}$$
$$\vec{M}_C^S = rR_3\vec{e}_2 - rR_2\vec{e}_3.$$

Nonholonomic constraints are obtained from the condition that there is no slip at the contact point:

$$\vec{v}_C = \vec{\omega} \times (\vec{r}_C - \vec{r}_P) \quad \Rightarrow \quad L\dot{\varphi}\cos\theta = r\omega_3 \ \wedge\ L\dot{\theta} = -r\omega_2, \tag{4}$$

where $\omega_i, i = 1,2,3$ are projections of the ball's angular velocity onto the movable coordinate system at point C (Fig. 4).
Now, dynamic equations can be written in the form:

$$-\frac{mr^2}{L}(\omega_2^2 + \omega_3^2) = N - mg\sin\theta\,,$$
$$m(r\dot{\omega}_3 + r^2\omega_2\omega_3\tan\theta/L) = R_2 + F_2,$$
$$m(-r\dot{\omega}_2 + r^2\omega_3^2\tan\theta/L) = R_3 + F_3 - mg\cos\theta, \tag{5}$$
$$J\dot{\omega}_1 = 0,$$
$$J(\dot{\omega}_2 - r\omega_3^2\tan\theta/L - r\omega_1\omega_3/L) = rR_3,$$
$$J(\dot{\omega}_3 + r\omega_2\omega_3\tan\theta/L + r\omega_1\omega_2/L) = -rR_2.$$

The total power of control forces in generalized brachistochrone problems equals zero [11]:

$$F_2 v_{C2} + F_3 v_{C3} = 0 \ \ \Rightarrow \ \ F_2\omega_3 - F_3\omega_2 = 0, \tag{6}$$

so that from dynamic equations (5) the conservation of the total mechanical energy follows

$$\frac{1}{2}J\omega_1^2 + \frac{1}{2}(J + mr^2)(\omega_2^2 + \omega_3^2) + mgL\sin\theta = E. \tag{7}$$

as well as the maintenance of the projection of the ball's angular velocity onto the radial direction:

$$\omega_1 = C_1. \tag{8}$$

By introducing dimensionless variables:

$$r' = r/r = 1,\ \ L' = L/r = 3,\ \ t' = t\sqrt{\frac{g}{r}}\,,\ \ \omega_i' = \omega_i\sqrt{\frac{r}{g}},$$
$$N' = N/(mg), F_i' = F_i/(mg)\,,\ \ R_i' = R_i/(mg)\,,\ \ E' = E/(mg). \tag{9}$$

in the text below, the label "prim" will be removed, and all expressions will be in dimensionless variables.

The laws of change in the control forces are of the dimensionless form:

$$F_2 = \frac{2}{15}\omega_1\omega_2 + \frac{7}{15}\omega_2\omega_3 \tan\theta + \frac{7}{5}\dot{\omega}_3,$$
$$F_3 = \frac{2}{15}\omega_1\omega_3 + \frac{7}{15}\omega_3^2 \tan\theta - \frac{7}{5}\dot{\omega}_2 + \cos\theta\,, \tag{10}$$

and the reactions of constraints are:

$$N = \sin\theta - \frac{1}{3}(\omega_2^2 + \omega_3^2) = \frac{17}{7}\sin\theta - \frac{10}{21}(E - C_1^2/5),$$
$$R_2 = -\frac{2}{15}(\omega_1\omega_2 + \omega_2\omega_3 \tan\theta + 3\dot{\omega}_3), \tag{11}$$
$$R_3 = -\frac{2}{15}(\omega_1\omega_3 + \omega_3^2 \tan\theta - 3\dot{\omega}_2).$$

Detachment angle $(N(\theta_{perf.rough}) = 0)$for the case of perfect rough sphere and zero initial value of the angular velocity projection ω_1 onto the radial direction:$\sin(\theta_{perf.rough}) = \frac{10E}{51}$.

Condition for non-slip occurrence based on Coulomb's laws of sliding friction:

$$R = \sqrt{R_2^2 + R_3^2} \leq \mu N. \tag{12}$$

The next section considers the brachistochrone problem as a problem of optimal control when the sphere is perfectly rough and there are no restrictions (12), and section 3 deals with the case of a real rough sphere, when given restriction must be taken into account.

3. The brachistochrone problem for the case of perfect rough sphere

Let us formulate the problem of optimal control that will be solved using Pontryagin's maximum principle [12].
Let it be known at the initial moment:

$$t_0 = 0,\ \varphi(t_0) = 0,\ \theta(t_0) = \theta_0,\ E(t_0) = E, \tag{13}$$

and let it be known at the final moment:

$$t_f = ?,\ \varphi(t_f) = \varphi_f,\ \theta(t_f) = \theta_f, \tag{14}$$

where:

$$\theta_{perf.rough} < \theta_f \leq \theta_0 < \frac{\pi}{2},\ 0 \leq \varphi_f < \pi. \tag{15}$$

Differential equations of this problem of optimal control can be obtained from (4), where it has been taken that $L = 3$:

$$\dot{\psi} = \omega_1,\ \dot{\varphi} = \frac{\omega_3}{3\cos\theta},\ \dot{\theta} = \frac{-\omega_2}{3}. \tag{16}$$

Energy integral (7) obtains the form:

$$2\omega_1^2 + 7(\omega_2^2 + \omega_3^2) + 30\sin\theta = 10E. \tag{17}$$

The optimal control problemis as follows: For the mechanical system, for specified differential equations(16) and initial (13) and end conditions (14), determine the motion of the system in minimum time while maintaining the energy integral (17).
In order to apply the maximum principle, cost functional is formed:

$$t_f = \int_{t_0}^{t_f} dt. \tag{18}$$

Pontryagin's function:

$$\begin{aligned} H &= -1 + \lambda_\psi\omega_1 + \lambda_\varphi\omega_3/(3\cos\theta) - \lambda_\theta\omega_2/3 + \\ &\quad + \rho(2\omega_1^2 + 7(\omega_2^2 + \omega_3^2) + 30\sin\theta - 10E), \end{aligned} \tag{19}$$

and costate system:

$$\dot{\lambda}_\psi = 0,\ \ \dot{\lambda}_\varphi = 0,\ \ \dot{\lambda}_\theta = -\lambda_\varphi\omega_3\sin\theta/(3\cos^2\theta) - 30\rho\cos\theta. \tag{20}$$

Transversality conditions are of the form:

$$\lambda_\psi(0) = 0,\ \ \lambda_\psi(t_f) = 0. \tag{21}$$

Maximum principle yields the conditions:

$$\begin{aligned} &\frac{\partial H}{\partial \omega_i} = 0 \quad \Rightarrow \quad \omega_1 = -\lambda_\psi/(4\rho)\,, \\ &\omega_2 = \lambda_\theta/(42\rho)\,,\ \omega_3 = -\lambda_\varphi/(42\rho\cos\theta)\,. \end{aligned} \tag{22}$$

The final moment is indefinite so that:

$$H = 0 \ \ \Rightarrow \ \ \rho = 1/(20(3\sin\theta - E)). \tag{23}$$

There is no rotation around the axis in the radial direction:

$$\left(\dot{\lambda}_\psi(t) = 0 \wedge \lambda_\psi(0) = 0 \wedge \lambda_\psi\left(t_f\right) = 0\right) \Rightarrow \lambda_\psi(t) = 0 \Rightarrow \omega_1(t) = 0. \tag{24}$$

On the brachistochronic motion, based on (11) and (20-24), the components of the constraint reaction are:

$$N = (51 \sin\theta - 10E)/21\,, \;\; R = \sqrt{R_2^2 + R_3^2} = 2\cos\theta/7. \tag{25}$$

When the ball is rolling down, the angle is decreasing, the normal component of the constraint reaction N is decreasing too, whereas tangential component R is increasing, so that in term of the slip, the critical slip is at the end of motion.
Now, the discussion on the possible values of the task parameters θ_f, E, μ can be conducted, where $\theta_0 = 1.5$

1. $N \geq 0 \Rightarrow E \leq 51\sin\theta_f/10$ the point must be below the red colored surface in Fig. 5;
2. $R \leq \mu N \Rightarrow E \leq 3(17\mu\sin\theta_f - 2\cos\theta_f)/(10\mu)$ the point must be below the blue colored surface;
3. $E \geq 3\sin\theta_0$ the point must be above the green colored surface because the initial kinetic energy is non-negative

This means that the representative point (θ_f, μ, E) corresponding to the task parameters must be located simultaneously above the green and below blue surface. The red surface is always above the blue one, which means that if there is no slip at the end point, there will be no detachment either.

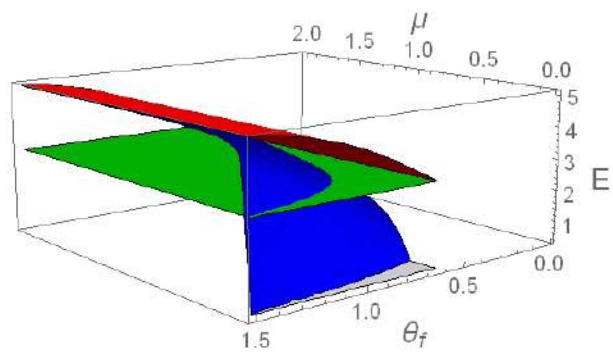


Fig. 5 Discussion on the possible positions of the representative point(θ_f, μ, E)

Also, it is noticeable that identical value R of the constraint reaction in the tangential plane is obtained from (10) and (11) when the motion is non-controlled $F_2 = F_3 = 0$ and when $\omega_1 = 0$, so the above discussion is applicable to non-controlled free rolling of a heavy ball along a real rough sphere.
Numerical solution parameters for the case of perfect rough surface in this example is:

$$\theta_0 = 1.5,\;\; E = 3,\;\; \theta_f = \pi/4\,,\;\; \varphi_f = \pi/2\,. \tag{26}$$

Two-point boundary value problem of the maximum principle, in this case, has differential equations:

$$\begin{aligned}
&\dot{\varphi} = 10\lambda_\varphi(1 - \sin\theta)/(21\cos^2\theta)\,,\\
&\dot{\theta} = 10\lambda_\theta(1 - \sin\theta)/21\,,\\
&\dot{\lambda}_\varphi = 0,\\
&\dot{\lambda}_\theta = -10\lambda_\varphi^2(1 - \sin\theta)\sin\theta/(21\cos^3\theta) + \cos\theta/(2(1 - \sin\theta))\,,
\end{aligned} \tag{27}$$

with initial conditions:

$$\begin{aligned}
&t_0 = 0,\ \varphi(t_0) = 0,\ \theta(t_0) = \theta_0,\ \lambda_\varphi(t_0) = ?\,,\\
&H(t_0) = 0 \Rightarrow \lambda_\theta(t_0) = -\sqrt{21/\left(10(1 - \sin\theta_0)\right) - \lambda_\varphi^2(t_0)/\cos^2\theta_0}\,.
\end{aligned} \tag{28}$$

We're choosing $\lambda_\varphi(t_0)$, t_f and shooting: $\varphi(t_f) = \varphi_f$, $\theta(t_f) = \theta_f$ where $\lambda_\varphi^2(t_0) \le 21\cos^2\theta_0/(10(1-\sin\theta_0)), t_f > 0$ Numerical solution this two-point boundary value problem is: $t_f = 5.882531$, $\lambda_\varphi(0) = 1.08448$ and diagrams of the law of change in polar angles and required coefficient of friction $\mu = R/N = 2\cos\theta/(17\sin\theta - 10)$ are given in Fig. 6.

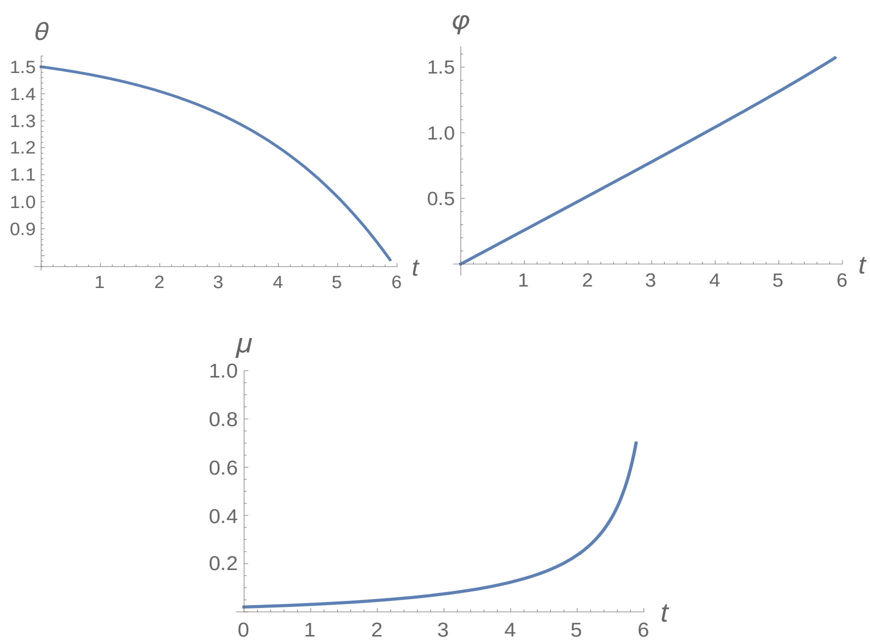


Fig. 6 Brachistochronic rolling of a ball along a perfect rough surface

A new form of differential equations (27):

$$\frac{d\varphi}{d\theta}=\frac{\lambda_\varphi}{\lambda_\theta\cos^2\theta}=\frac{\lambda_\varphi}{-\cos^2\theta\sqrt{21/(10(1-\sin\theta))-\lambda_\varphi^2/\cos^2\theta}},$$
$$\frac{dt}{d\theta}=\frac{-3}{\omega_2}=\frac{-21}{10\lambda_\theta(\sin\theta-1)}=\frac{-21}{10(-\sqrt{21/(10(1-\sin\theta))-\lambda_\varphi^2/\cos^2\theta})(\sin\theta-1)}, \tag{29}$$

gives us that the solution can be also reached through the squares, given that the angle θ is monotonically decreasing over time:

$$\varphi_f=\int_{\theta_f}^{\theta_0}\frac{\lambda_\varphi(0)d\theta}{\cos^2\theta\sqrt{21/(10(1-\sin\theta))-\lambda_\varphi^2(0)/\cos^2\theta}},$$
$$t_f=\int_{\theta_f}^{\theta_0}\frac{21d\theta}{10\sqrt{21/(10(1-\sin\theta))-\lambda_\varphi^2(0)/\cos^2\theta}(1-\sin\theta)}. \tag{30}$$

By solving integral equations (30), it is easier to obtain the already obtained numerical solutions for $(t_f,\lambda_\varphi(0))$.
Necessary value of the Coulomb coefficient of friction at the beginning of motion is:

$$\mu^*=R(0)/N(0)=2\cos\theta_0/(17\sin\theta_0-10)=0.0203343\,, \tag{31}$$

whereas necessary value of the Coulomb coefficient of friction at the end of motion is:

$$\mu^{**}=R(t_f)/N(t_f)=2\cos\theta_f/(17\sin\theta_f-10)=0.699823. \tag{32}$$

The assumption that surfaces are perfectly rough in this task of the brachistochronic motion is satisfied only for $\mu\geq\mu^{**}$ (e.g., rubber on rubber and glass on glass). If this is not the case (e.g., wood on wood), slip on the brachistochronic motion would occur earlier. The analysis also holds for non-controlled motion of a heavy ball.

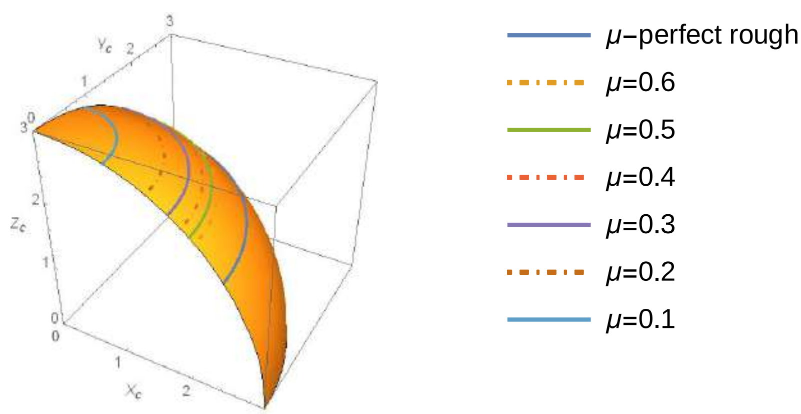


Fig. 7 Slip circles depending on the Coulomb coefficient of friction

Figure 7 shows slip circles that yield the lowest possible end positions during brachistochronic motion, depending on the coefficient of friction. This means that the lower the coefficient of friction, the more restricted the region where the surface is considered perfectly rough. It may even happen that for $\mu < \mu^*$ slip occurs at the beginning of motion. In the case when $\mu^* \leq \mu < \mu^{**}$ the problem of optimal control should include the restriction (12) and the task of optimal control becomes considerably complicated. Such possibility will be analyzed in the section below.

4. The brachistochrone problem for the case of imperfect rough sphere ($\mu^* \leq \mu < \mu^{**}$)

Let us observe the brachistochrone problem with numerical parameters (26) and seek the solutions in a neighborhood of $\mu = \mu^{**}$ where over the entire interval:

$$\omega_2(t) > 0,\ \dot{\omega}_2(t) > 0,\ \omega_3(t) = \sqrt{30\,(1 - \sin\theta(t))/7 - \omega_2^2(t)} > 0. \tag{33}$$

Differential equations written through the theta angle as independent variables are:

$$\begin{aligned} &\frac{dt}{d\theta} = \frac{-3}{\omega_2}, \\ &\frac{d\varphi}{d\theta} = -\sqrt{30(1-\sin\theta)/7 - \omega_2^2}\Big/(\omega_2 \cos\theta), \\ &\frac{d\omega_2}{d\theta} = \frac{-3}{\omega_2}\Big((30(1-\sin\theta)/7 - \omega_2^2)\tan\theta/3 + 5R_3/2\Big). \end{aligned} \tag{34}$$

Determination of the set of permissible values of the control $u = R_3$ is based on (5), (6) and (12):

$$\begin{matrix} R_3\omega_2 - R_2\omega_3 = 2\omega_2\cos\theta/7 \\ R_2^2 + R_3^2 \leq \mu^2 N^2 \end{matrix} \Rightarrow$$

$$\begin{aligned} &R_3 \in \left[R_3^{min}, R_3^{max}\right], \\ &R_3^{max} = \frac{2\cos\theta\,\omega_2^2 + \omega_3\sqrt{49\mu^2N^2(\omega_2^2+\omega_3^2) - 4\omega_2^2\cos^2\theta}}{7(\omega_2^2+\omega_3^2)}, \\ &R_3^{min} = \frac{2\cos\theta\,\omega_2^2 - \omega_3\sqrt{49\mu^2N^2(\omega_2^2+\omega_3^2) - 4\omega_2^2\cos^2\theta}}{7(\omega_2^2+\omega_3^2)}. \end{aligned} \tag{35}$$

In order that the set of permissible controls will not be an empty set:

$$\omega_2^2 \leq \frac{49\mu^2N^2(\omega_2^2+\omega_3^2)}{4\cos^2\theta} = \frac{30\mu^2(17\sin\theta - 10)^2(1-\sin\theta)}{28\cos^2\theta}. \tag{36}$$

For the solutions in a neighborhood of $\mu = \mu^{**}$ it is sufficient to introduce the restriction:

$$\omega_2(t_f) \le \mu(17\sin\theta_f - 10)\sqrt{30(1-\sin\theta_f)/28}/\cos\theta_f. \tag{37}$$

Let us assume that this condition is satisfied in order to avoid a very complex task of optimal control. It will be checked only subsequently after numerical solution is obtained.

Cost functional in this new problem of time minimization is:

$$t_f = \int_{\theta_1}^{\theta_0} \frac{3d\theta}{\omega_2}. \tag{38}$$

whereas Pontryagin's function

$$\begin{aligned} H = &\frac{-3}{\omega_2} - \frac{\lambda_\varphi\sqrt{30(1-\sin\theta)/7-\omega_2^2}}{\omega_2\cos\theta} - \\ &-\frac{3\lambda_{\omega 2}}{\omega_2}\left((30\,(1-\sin\theta)/7-\omega_2^2)\tan\theta/3+\frac{5u}{2}\right)+ \\ &+\rho(u^2+\frac{\omega_2^2(u-2\cos\theta/7)^2}{30(1-\sin\theta)/7-\omega_2^2}-\frac{\mu^2(17\sin\theta-10)^2}{49}). \end{aligned} \tag{39}$$

Here, the maximum principle gives the following possibilities for optimal control over some of the intervals

$$\begin{aligned} u_{opt} &= \begin{cases} u_s,\ \lambda_{\omega 2}=0 \\ R_3^{max},\ \ \lambda_{\omega 2}<0 \\ R_3^{min},\ \ \lambda_{\omega 2}>0 \end{cases} \\ \rho &= \begin{cases} 0,\ \lambda_{\omega 2}=0 \\ \frac{15\lambda_{\omega 2}(7\omega_2^2-30+30\sin\theta)}{8\omega_2(\omega_2^2\cos\theta+15R_3^{max}(-1+\sin\theta))},\ \lambda_{\omega 2}<0 \\ \frac{15\lambda_{\omega 2}(7\omega_2^2-30+30\sin\theta)}{8\omega_2(\omega_2^2\cos\theta+15R_3^{min}(-1+\sin\theta))}.\ \lambda_{\omega 2}>0 \end{cases} \end{aligned} \tag{40}$$

The costate variable $\lambda_{\omega 2}$ in this task of optimal control has a role of so-called "switching function", so that its sign on a non-singular part determines whether the control will be on the upper or lower limit. On a singular part it equals zero.

Calculation of the singular optimal control [13] can be performed in the following manner:

$$\begin{aligned} &\lambda_{\omega 2}=0 \Rightarrow \frac{d\lambda_{\omega 2}}{d\theta} = -\frac{\partial H}{\partial\omega_2}=0 \Rightarrow\ \lambda_\varphi = -\frac{\sqrt{7}\cos\theta\sqrt{30(1-\sin\theta)-7\omega_2^2}}{10(1-\sin\theta)}, \\ &\frac{d^2\lambda_{\omega 2}}{d\theta^2}=0 \Rightarrow u_s=\frac{2}{7}\cos\theta\,\frac{7\omega_2^2+15\sin\theta-15}{15(\sin\theta-1)} \Rightarrow R=\frac{2}{7}\cos\theta\,. \end{aligned} \tag{41}$$

On a non-singular part:

$$R = \mu N = \mu(17\sin\theta - 10)/7. \tag{42}$$

The assumed structure of the optimal control in a neighborhood of $\mu = \mu^{**}$ is:

$$u_{opt}=\begin{cases} u_s, & \theta_0 \geq \theta > \theta^* \\ R_3^{min}, & \theta^* \geq \theta \leq \theta_f \end{cases}. \tag{43}$$

Numerical solution procedure (two-parameter shooting) consists of the following steps:

1. by choosing $\omega_2(\theta_0)$ and θ^*, numerical integration of the basic system over the interval$[\theta_0, \theta^*]$

$$\begin{aligned} \frac{d\varphi}{d\theta} &= -\sqrt{30(1-\sin\theta)/7-\omega_2^2}/(\omega_2\cos\theta), \quad \varphi(\theta_0)=0 \\ \frac{d\omega_2}{d\theta} &= -\frac{3}{\omega_2}\Big((30\,(1-\sin\theta)/7-\omega_2^2)\tan\theta/3+5u_s/2\Big). \end{aligned} \tag{44}$$

2. and basic and costate system over the interval $[\theta^*, \theta_f]$

$$\begin{aligned} \frac{d\varphi}{d\theta} &= -\sqrt{30(1-\sin\theta)/7-\omega_2^2}/(\omega_2\cos\theta), \\ \frac{d\omega_2}{d\theta} &= -\frac{3}{\omega_2}\Big((30\,(1-\sin\theta)/7-\omega_2^2)\tan\theta/3+5R_3^{min}/2\Big), \\ \frac{d\lambda_\varphi}{d\theta} &= 0, \qquad \lambda_\varphi(\theta^*) = -\frac{\sqrt{7}\cos\theta^*\sqrt{30(1-\sin\theta^*)-7\omega_2^2(\theta^*)}}{10(1-\sin\theta^*)}, \\ \frac{d\lambda_{\omega 2}}{d\theta} &= -\frac{\partial H}{\partial\omega_2} = \cdots, \qquad \lambda_{\omega 2}(\theta^*) = 0. \end{aligned} \tag{45}$$

fulfillment of conditions is $\varphi(\theta_f)=\varphi_f$, $\lambda_{\omega 2}(\theta_f)=0$ ensured.
Numerical example was done for the value of Coulomb's coefficient $\mu = 0.6$ and numerical solutions were obtained $\theta^* = 0.821251$, $\omega_2(\theta_0) = 0.0878961$.

The diagram of the "switching function" is given in Fig. 8, the dashed line ($\mu = 0.6$). The final part of the motion is shown only, the solid line indicating the case when it is possible to a have a singular part over the entire interval ($\mu \geq \mu^{**}$), and the dash-dotted line designating a boundary case ($\mu = \bar{\mu}$).

By gradually decreasing the coefficient of friction and conducting the numerical solution procedure, it can be established that conditions (37) are disrupted for $\bar{\mu} = 0.576383$.

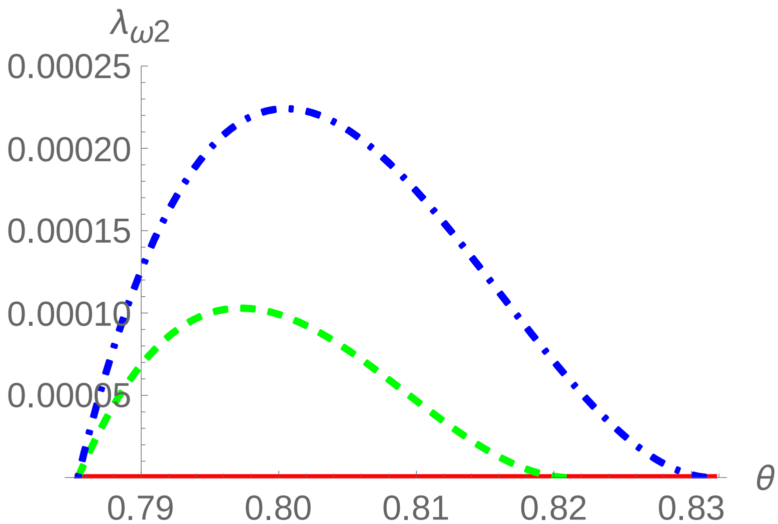


Fig. 8 The switching function $\lambda_{\omega 2}$

Here, it is also necessary to check whether the conditions (40) of the maximum principle are fulfilled after numerical solutions is done

$$\lambda_{\omega 2}(\theta) \geq 0, \quad \theta^* \leq \theta \leq \theta_f, \tag{46}$$

as well as the condition (37).
If the coefficient of friction is lower than $\bar{\mu}$ the structure of optimal control (43) changes, and the numerical solution procedure becomes more complex.

5. Conclusions

The problem of brachistochronic motion of a heavy uniform ball rolling without slip along the upper outside surface of an imperfect rough stationary sphere is solved.
In the first part of the paper the determination of the brachistochronic motion is solved as the problem of optimal control using Pontryagin's maximum principle. Three projections of the ball's angular velocity onto the base vectors of the spherical coordinate system are taken for controls. The two-point boundary value problem, which is reduced to the two-parameter shooting of one coordinate of the conjugate vector and end moment, is solved. It is shown that there is no angular velocity projection onto the radial direction. This solution corresponds to the motion of the heavy ball along a perfect rough sphere, because it is necessary to ensure unrealistically high Coulomb coefficient of sliding friction.
It is shown that mutual detachment of the bodies cannot occur before their mutual slipping at the contact point. A corresponding numerical example is given, with graphical representation of the effects of initial energy values, Coulomb coefficient and ultimate height of the ball center on the solution

structure, in this case. The review highlights regions where it is possible to obtain a singular control across the entire motion or a combination of a singular and non-singular part of the optimal trajectory.

The second part provides the following discussion: if the constraint between the sphere and the ball is imperfectly rough, the formulation of the optimal control problem should include restrictions to the ratio between the tangential and the normal components of the reaction of constraint. Here, the problem of optimal control is formulated in such way that the tangential component of the reaction of constraint is taken for the control, with the restriction resulting from Coulomb laws of sliding friction. The problem thus formulated belongs to the theory of singular optimal controls, and the solution that satisfies the Maximum principle consists of a singular part at the beginning of motion and a non-singular part, during which the ratio between mentioned components has maximum possible value that concrete surfaces can achieve.

Futher research of this problem involves: research on the structure of control and appropriate numerical solutions for $\mu^* < \mu < \bar{\mu}$, generalization of the result to the ball rolling on the stationary rotating surface and generalization of the result to the ball rolling on the moving surface which is rotating about vertical axis at constant angular velocity.

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