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# OPTIMAL CONTROL OF MECHANICAL SYSTEM MOTION WITH LIMITED REACTIONS OF CONSTRAINTS

**J.Vuković[1],A.Obradović[1], Z. Mitrović[1]**

[1] Faculty of Mechanical Engineering,
The University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35
e-mail: aobradovic@mas.bg.ac.rs

Abstract. The problem of determining the mechanical system limited control forces, with simultaneously imposed limitations to reactions of constraints is solved. Both external and internal constraints that can be holonomic or nonholonomic are analyzed. In this regard, differential equations of motion are formed with explicitly present Lagrange's multipliers of the constraints. Phase space dimensions and structure depend of the constraints that are of interest for the present problem. Dependence of multipliers of the constraints on corresponding reactions of constraints intensity is established. Control forces limitations and reactions of constraints intensity are given in the form of inequations. The problem of optimal control is solved by Pontryagin's Maximum Principle. The proposed method has been applied to two examples of motion in a minimal time between two specified positions. The first example provides solution for the case of the motion of a material point along holonomic constraint of limited reaction. The second example refers to solution for the case of a rigid body with nonholonomic constraint – by the Chaplygin blade type, whose reaction of constraint is limited in intensity.

## 1. Introduction

Methods of analytical mechanics [1] have proved their efficiency in forming a minimal number of differential equations of mechanical systems motion. In the case of holonomic systems with ideal constraints, Lagrange's equations of the second kind enable the determination of the system motion without considering reactions of the constraints. In the systems where consideration of reactions of the constraints is inevitable [2], the multipliers of the constraints occur in differential equations. They are then solved together with equations of constraints. In some problems of the control of mechanical system motion, due to the avoidance of impermissible loads or for practical reasons, there are requirements to limit the reactions of the system constraints. As the directions of reactions are determined by the constraints, those requirements refer to limiting the projections of reactions of the constraints to their eigendirections. In that case it would be desirable that those reactions of the constraints explicitly figure in differential equations of motion. In the present paper the equations of constraints are transformed into such form that their multipliers are the very projections of the corresponding reactions of the constraints to their eigendirections. Limitations to reactions of the constraints impose additional limitations to control forces by whose action the controlled motion is realized. In the considered problem the choice of control and optimality criterion influence the Pontryagin's function [3] that it has the form which imposes compulsory examination of singular controls [4] phenomenon. To illustrate the proposed method, two examples have been solved. The first example refers to the optimal control of a material particle along unilateral holonomic constraint, with limited reactions. In the second example the time of motion of a rigid body, whose motion is subject to nonholonomic constraint with limited reactions, is minimized.

## 2. Optimal control with limited reactions

Let us consider the controlled motion of a scleronomic mechanical system with $n$ degrees of freedom in a space whose configuration is determined by the $n+l$ system of generalized coordinates $q^i$ with the presence of $l$ ideal independent stationary constraints, among which there are all of $r$ nonholonomic constraints:

$$h_i^\mu \dot{q}^i = 0, \; h_i^\mu = h_i^\mu(q), \; i = 1,...,m+l, \; \mu = 1,...,r \tag{1}$$

and $s$ holonomic constraints:

$$f^\nu(q) = 0 \;, \; \nu = 1,...,s \tag{2}$$

whose consideration is imposed by the conditions of the problem. Differentiation of holonomic constraints with respect to time is reduced to differential form so that all $l = r + s$ constraints (1) and (2) are formally expressed by equations:

$$d_i^\alpha \dot{q}^i = 0, \; d_i^\alpha = d_i^\alpha(q), \; \alpha = 1,...,l \tag{3}$$

The quantities $d_i^\alpha$ are covariant coordinates of the vector $\vec{d}^\alpha$ that, in the case of ideal constraints, determine the directions of corresponding reactions $\vec{R}_\alpha$. If the norm $\left\|d^\alpha\right\|$ is defined in the configuration space, which presents the intensity of vector $\vec{d}^\alpha$, equations (3) can be transformed into the form:

$$b_i^\alpha \dot{q}^i = 0, \; b_i^\alpha = d_i^\alpha / \left\|d^\alpha\right\|, \; \left\|b^\alpha\right\| = 1, \tag{4}$$

where the quantities $b_i^\alpha(q)$ present the covariant coordinates of the unit vector $\vec{b}^\alpha$ in the direction of the reaction $\vec{R}_\alpha$. Thus, the multipliers of the constraints in expressions for generalized reactions:

$$Q_i^R = \lambda_\alpha b_i^\alpha \tag{5}$$

present the projections $R_\alpha$ of the $\vec{R}_\alpha$ constraints reactions to their eigendirection, whose orientation is determined by the unit vector $\vec{b}^\alpha$ i.e. : $\lambda_\alpha = R_\alpha$ .

Transformation of the constraints to the form (4) enabled the direct introduction into consideration the reactions with required limitations:

$$R'_\alpha \leq R_\alpha \leq R''_\alpha \tag{6}$$

Let the control of mechanical system motion be realized by the action of $m$ active control forces:

$$\vec{F}_p^u = F_{(p)}^u \vec{e}^{(p)} \ , \ \left|\vec{e}^{(p)}\right| = 1 \ , \ p = 1,...,m, \tag{7}$$

with limitations:

$$F_p^{u'} \geq F_p^u \leq F_p^{u''} \tag{8}$$

where the quantities $F_p^u$ present the projections of forces $\vec{F}_p^u$ onto eigendirections, whose orientation is determined by the unit vectors $\vec{e}^p$. The corresponding generalized control forces are:

$$Q_i^u = c_i^p F_p^u \tag{9}$$

where:

$$c_i^p(q) = \vec{e}^{(p)} \frac{\partial \vec{r}_{(p)}}{\partial q^i}. \tag{10}$$

Let, besides the control forces $Q_i^u$ and reactions of the constraints $Q_i^R$, the system be subjected to the action of some specified forces:

$$Q_i = Q_i(q, \dot{q}) \tag{11}$$

then, to consider its controlled motion in the configuration, countervariant equations [1] may be used

$$\ddot{q}^j + \Gamma_{ki}^j \dot{q}^j \dot{q}^k = a^{ij}\left(Q_i + b_i^\alpha R_\alpha + c_i^p F_p^u\right), \ j,k = 1,...,m+l, \tag{12}$$

where: $a^{ij}$ is countervariant metric tensor and $\Gamma^j_{ki}$ are the corresponding Cristoffel's symbols. Equations (12) should be considered together with equations of constraints (4), by whose differentiation with respect to time one obtains $l$ conditions for the second derivatives of generalized coordinates:

$$b_i^\alpha \ddot{q}^i + \frac{\partial b_i^\alpha}{\partial q^j} \dot{q}^i \dot{q}^j = 0, \tag{13}$$

By eliminating the second derivatives $\ddot{q}^i$ from equations (12) and (13) one obtains $l$ relation between reactions of the constraints and control forces:

$$R_\beta = \psi_\beta + \gamma_\beta^p F_p^u \ , \ \beta = 1,...,l, \tag{14}$$

where:

$$\psi_\beta = h_{\alpha\beta}\left( -\frac{\partial b_i^\alpha}{\partial q^j} \dot{q}^i \dot{q}^j - b_j^\alpha \left( -\Gamma^j_{ki} \dot{q}^k \dot{q}^i + a^{ij} Q_i \right)\right),$$
$$h^{\alpha\beta} = b_j^\alpha a^{ij} b_i^\beta, \qquad \gamma_\beta^p = -h_{\alpha\beta} b_j^\alpha a^{ij} c_i^p, \tag{15}$$

### 3. The problem of optimal control

Further procedure involves the definition of the problem of optimal control of the considered system motion. In accordance with the methods of control theory [3], the system motion is considered in the state space characterized by variables $q^i, y^i = \dot{q}^i$. By substituting (14) into equations (12) and by introducing variable states, one obtains $2n + 2l$ differential equations of the first order:

$$\dot{q}^j = y^j$$
$$\dot{y}^j = D^j(q, y) + E^{jp}(q) u_p \tag{16}$$

where:

$$D^j = \psi_\beta a^{ij} b_i^\beta - \Gamma^j_{ki} \dot{q}^k \dot{q}^i + a^{ij} Q_i, E^{jp} = a^{ij}\left(c_i^p + b_i^\alpha \gamma_\alpha^p\right) \tag{17}$$

and where controls $u_p$ present the projections of control forces onto eigendirections i.e.

$$u_p = F_p^u. \tag{18}$$

With regard to (14), in addition to limitations (8), the limitations (6) should be taken into account so that permissible controls belong to the intersection of sets:

$$G_u^1 : F_p^{u'} \geq u_p \leq F_p^{u''}; \qquad G_u^2 : R_\beta' \geq \psi_\beta + \gamma_\beta^p u_p \leq R_\beta'', \\ u_p \in G_u = G_u^1 \cap G_u^2. \tag{19}$$

The problem of optimal control is the determination of controls $u_p^*$ and their corresponding trajectories among the permissible controls $u_p$ (19) so that the system whose motion is described by equations (16) moves from the initial position:

$$t_0;\ \theta_\sigma^0(q(t_0), y(t_0)) = 0\ ,\ \ \sigma = 1,\ldots,m_0 \leq 2n + 2l \tag{20}$$

to the final position:

$$t_1;\ \theta_\rho^1(q(t_1), y(t_1)) = 0\ ,\ \ \rho = 1,\ldots,m_1 \leq 2n + 2l \tag{21}$$

under the condition:

$$\int_{t_0}^{t_1} f_0(q, y)dt \xrightarrow[u \in G_u]{} \inf\ . \tag{22}$$

The problem thus defined has been adapted for direct application of the maximum principle [3]. It should be mentioned that this method can be applied to define the problem of optimal control and, in a more general case, to all scleronomic mechanical systems with ideal constraints, whose reactions are subjected to limitations (6).

### 3. Singular controls

In this paper the choice of control in the form (18) and functionals (22) causes the Pontryagin's function of this problem to have the form:

$$H(q, y, u, \lambda) = H_0 + H^p u_p \tag{23}$$

where:

$$H_0 = \lambda_0 f_0(q,y) + \lambda_j y^j + \lambda_{n+l+j} D^j(q,y),$$
$$H^p = \lambda_{n+l+j} E^{jp}(q). \tag{24}$$

The quantities $\lambda_0, \lambda_s, s = 1,...,2n+2l$, form the conjugate vector [3], where constant $\lambda_0 \le 0$. The Pontryagin's function (23) linear in control $u_p$ indicates that it is inevitable to examine the singular controls [4] phenomenon. The maximum principle provides necessary conditions for determination of singular controls. If among the permissible controls, on subintervals of the time interval $[t_0, t_1]$, there are singular controls, to make them optimal too, it is necessary for them to satisfy the corresponding Kelly's conditions [5]. In the general case, the system trajectory in the state space consists of singular and non-singular parts in some individual controls. The entire time interval $[t_0, t_1]$ is divided into the corresponding subintervals for whose boundaries the necessary conditions [6] for the optimality of junctions between singular and nonsingular subarcs hold.

## 4. **Example1: Optimal motion of a material particle along the constraint with limited reaction**

Analysis is conducted of the motion of a material particle in a vertical plane along the ideal constraint $2ay - x^2 = 0$ (Fig.1), which, in accordance with transformation into the differential form (4), can be expressed in the form:

$$\frac{-x\dot{x} + a\dot{y}}{\sqrt{x^2 + a^2}} = 0, \tag{25}$$

so that the corresponding differential equations (12) are:

$$\ddot{x} = \frac{F}{m} - \frac{x}{\sqrt{x^2 + a^2}}\frac{R}{m},$$
$$\ddot{y} = \frac{a}{\sqrt{x^2 + a^2}}\frac{R}{m} - g \tag{26}$$

where $R$ is the projection of the reaction $\vec{R}$ onto the direction of the constraint gradient and presents its multiplier in equations (26). According to (14) its value is:

$$R = \frac{m(\dot{x}^2 + ga) + xF}{\sqrt{x^2 + a^2}}. \tag{27}$$

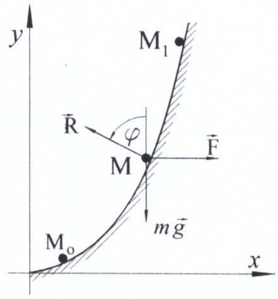


**Figure 1.** Material particle in a vertical plane

It is necessary to determine optimal control $F(t)$ that translates a material particle of mass m from the state of rest in the position $M_0(a/2, a/8)$ to the state of rest in the position $M_1(2a, 2a)$ for a specified time $t_1 = 1\,s$, under the condition for optimality:

$$\int_{t_0}^{t_1} \left(\dot{x}^2 + \dot{y}^2\right) dt \longrightarrow \inf \; . \tag{28}$$

and with limitation of reaction of the unilateral constraint:

$$0 \le \frac{R}{m} \le 25 \frac{m}{s^2}. \tag{29}$$

By introducing shift:

$$y' = tg\varphi = \frac{x}{a}, \quad v = \sqrt{\dot{x}^2 + \dot{y}^2}, \; u = \frac{F}{m}, \tag{30}$$

the problem of optimal control can be formulated in a simpler way:

$$\begin{aligned}
&\dot{\varphi} = \frac{v}{a}\cos^3\varphi \\
&\dot{v} = u\cos\varphi - g\sin\varphi, \\
&\int_0^{t_1} v^2 dt \to \inf, \\
&0 \le u\sin\varphi + g\cos\varphi + \frac{v^2}{a}\cos^3\varphi \le 25\frac{m}{s^2}, \\
&t_0 = 0; \;\; \varphi(t_0) = arctg\frac{1}{2}, \;\; v(t_0) = 0, \\
&t_1 = 1s; \;\; \varphi(t_1) = arctg 2, \;\; v(t_1) = 0.
\end{aligned} \tag{31}$$

According to the corresponding Pontryagin's function [3]:

$$H = -v^2 + \lambda_1 \frac{v}{a}\cos^3\varphi + \lambda_2(u\cos\varphi - g\sin\varphi) \tag{32}$$

the system of equations is obtained:

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_1 \frac{3v}{a}\cos^2\varphi\sin\varphi + \lambda_2(u\sin\varphi + g\cos\varphi) \\ \dot{\lambda}_2 &= 2v - \frac{\lambda_1}{a}\cos^3\varphi. \end{aligned} \tag{33}$$

For the determination of the first-order singular control [4], if it exists, the conditions used are:

$$\frac{\partial H}{\partial u} = 0, \quad \frac{d}{dt}\frac{\partial H}{\partial u} = 0, \quad \frac{d^2}{dt^2}\frac{\partial H}{\partial u} = 0, \tag{34}$$

wherefrom, according to (32) and (33), one obtains:

$$\lambda_2 = 0,\ \lambda_1 = \frac{2va}{\cos^3\varphi},\ u = g\,tg\varphi,\ v = const. \tag{35}$$

Kelly's condition [5] being satisfied too:

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u} = 2\cos\varphi > 0. \tag{36}$$

When considered over the entire interval of the controlled motion, it follows from the maximum principle that:

$$u = \begin{cases} \left(25\dfrac{m}{s^2} - g\cos\varphi - \dfrac{v^2}{a}\cos^3\varphi\right)/\sin\varphi, & \lambda_2 > 0, \\ g\,tg\varphi, & \lambda_2 = 0, \\ \left(-g\cos\varphi - \dfrac{v^2}{a}\cos^3\varphi\right)/\sin\varphi, & \lambda_2 < 0. \end{cases} \tag{37}$$

Integrals (35) enable numerical solution not to involve integration of the costate system (33), but calculation of time moments $t' = 0.153270\,s$ and $t'' = 0.727505\,s$ by the shooting method, under satisfying end-conditions $\varphi(t_1) = arctg2, v(t_1) = 0$. After that, using (35), integration of the costate system (33) is done. Fig. 2 shows diagrams $v(t),\ u(t), \lambda_2(t)$ over the entire interval of the controlled motion (for $a = 1m$ ). Also, it is noted that the necessary conditions for the optimality of junctions between singular and nonsingular subarcs [4, 6] are satisfied, which for the first-order singular control allow interruptions in control at junctions points.

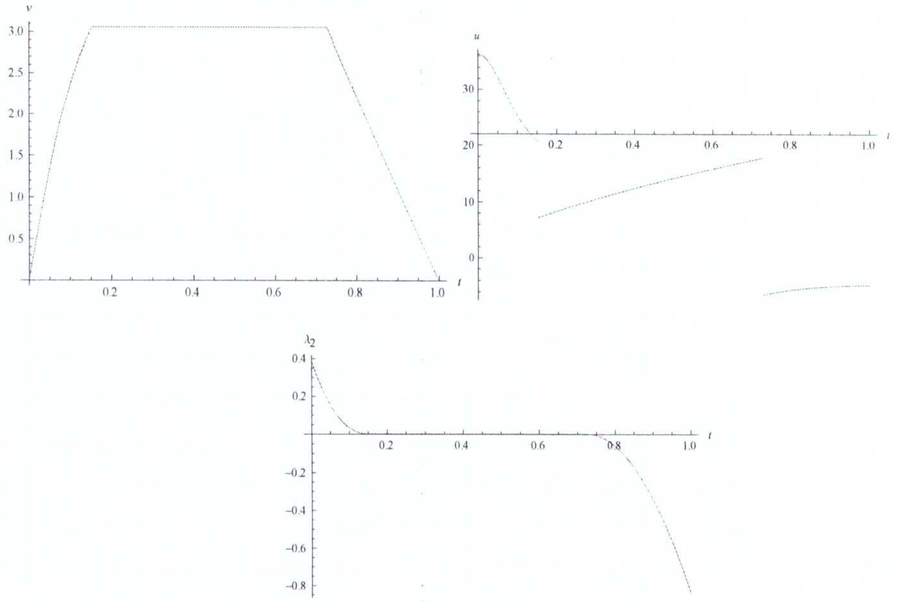


**Figure 2.** Diagrams $v(t),\ u(t), \lambda_2(t)$

### 5. Example2: Optimal control of a rigid body motion with limited reaction of nonholonomic constraint

Let us consider the motion of a rigid body in a horizontal plane, whose differential equations (12) are:

$$\ddot{x} = \frac{F}{m}\cos\varphi - \frac{R}{m}\sin\varphi$$
$$\ddot{y} = \frac{F}{m}\sin\varphi + \frac{R}{m}\cos\varphi, \tag{38}$$
$$\ddot{\varphi} = \frac{M}{J}$$

where the equation of nonholonomic constraint is written in the form (4):

$$\dot{y}\cos\varphi - \dot{x}\sin\varphi = 0. \tag{39}$$

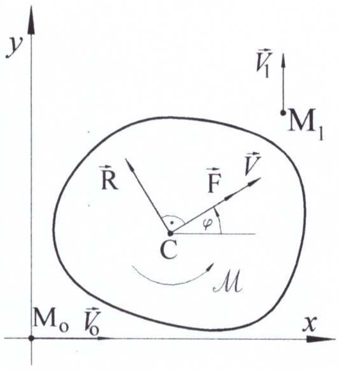


**Figure 3.** Rigid body motion in horizontal plane with nonholonomic constraints

Let us consider the problem of time minimization of a rigid body motion from the initial position:

$$t_0 = 0,\ x(t_0) = y(t_0) = 0,\ \dot{x}(t_0) = 1\frac{m}{s^2},\quad \varphi(t_0) = 0, \tag{40}$$

to the final position:

$$t_1 = ?,\ x(t_1) = y(t_1) = 10m,\ \dot{y}(t_1) = 1\frac{m}{s^2},\quad \varphi(t_1) = \frac{\pi}{2}, \tag{41}$$

where the control force, collinear with the center of inertia velocity, is limited in intensity:

$$\left|\frac{F}{m}\right| \le 1\frac{m}{s^2} \tag{42}$$

and the intensity of the constraint reaction is also limited:

$$\left|\frac{R}{m}\right| \le 1\frac{m}{s^2}. \tag{43}$$

The reaction has been determined according to (14):

$$R = m\dot{\varphi}(\dot{y}\sin\varphi + \dot{x}\cos\varphi) \tag{44}$$

and does not depend of control forces, so here we have the case of the control of system motion with a limited phase state:

$$|\dot{\varphi}(\dot{y}\sin\varphi + \dot{x}\cos\varphi)| \le 1\frac{m}{s^2}. \tag{45}$$

The fact that in this example neither $\dot{\varphi}(t_0)$ nor $\dot{\varphi}(t_1)$ is specified enables that by introducing:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2},\ u_1 = \frac{F}{m},\ u_2 = \frac{R}{m}, \tag{46}$$

this problem is simplified:

$$\begin{aligned}
&\dot{x} = v\cos\varphi \\
&\dot{y} = v\sin\varphi \\
&\dot{v} = u_1 \\
&\dot{\varphi} = \frac{u_2}{v} \\
&\int_0^{t_1} 1\,dt \to \inf, \\
&|u_1| \le 1\frac{m}{s^2}, \quad |u_2| \le 1\frac{m}{s^2}, \\
&t_0 = 0;\ x(t_0) = y(t_0) = 0, \quad \varphi(t_0) = 0,\ v(t_0) = 1\frac{m}{s^2}, \\
&t_1 = ?;\ x(t_1) = y(t_1) = 10m, \varphi(t_1) = \frac{\pi}{2},\ v(t_1) = 1\frac{m}{s^2}.
\end{aligned} \tag{47}$$

Pontryagin's function has the form:

$$H = -1 + \lambda_1 v\cos\varphi + \lambda_2 v\sin\varphi + \lambda_3 u_1 + \lambda_4 \frac{u_2}{v} \tag{48}$$

and the costate system:

$$\begin{aligned}
&\dot{\lambda}_1 = 0 \\
&\dot{\lambda}_2 = 0 \\
&\dot{\lambda}_3 = -\lambda_1 \cos\varphi - \lambda_2 \sin\varphi + \lambda_4 \frac{u_2}{v^2} \\
&\dot{\lambda}_4 = \lambda_1 v\sin\varphi - \lambda_2 v\cos\varphi
\end{aligned} \tag{49}$$

In this problem singular control for both controls:

$$\frac{\partial H}{\partial u_1} = \lambda_3 = 0, \ \frac{\partial H}{\partial u_2} = \frac{\lambda_4}{v} = 0, \tag{50}$$

leads to the conditions:

$$\lambda_1 = \lambda_2 = 0, \tag{51}$$

thereby to $H = -1$ , which is contrary to the maximum principle [3] for the case of non-specified $t_1$. Singular control $u_2$ is obtained from the condition:

$$\frac{\partial H}{\partial u_2} = 0, \ \frac{d}{dt}\frac{\partial H}{\partial u_2} = 0, \ \frac{d^2}{dt^2}\frac{\partial H}{\partial u_2} = 0, \tag{52}$$

wherefrom, on that part of the motion:

$$\lambda_4 = 0, \ tg\varphi = \frac{\lambda_2}{\lambda_1} = const, \ u_2 = 0, \tag{53}$$

and the Kelley condition [5] reduces to:

$$\frac{\partial}{\partial u_2}\frac{d^2}{dt^2}\frac{\partial H}{\partial u_2} = \lambda_1 \sin\varphi + \lambda_2 \cos\varphi > 0. \tag{54}$$

Let us look for optimal control of the following structure:

$$u_1 = \begin{cases} 1, & t \in [0, t'') \\ -1, & t \in [t'', t_1] \end{cases}, \quad u_2 = \begin{cases} 1, & t \in [0, t') \\ 0, & t \in [t', t'''] \\ -1, & t \in (t''', t_1] \end{cases}. \tag{55}$$

The system of differential equations (47), and according to (55), has analytical solutions wherefrom by satisfying the end-conditions (47) one obtains:

$$\begin{aligned} t' &= e^{\pi/4} - 1\, s, \\ t'' &= -1 + \sqrt{\left(102\sqrt{2} + 2e^{\pi/2}\right)/10}\ s, \\ t_1 &= 2t'', \ t''' = t_1 - t'. \end{aligned} \tag{56}$$

It is also necessary to integrate the costate system as well as to make sure that:

$$\lambda_3 \begin{cases} >0, & t \in [0, t'') \\ =0 & , t = t'' \\ <0, & t \in (t'', t_1] \end{cases}, \quad \lambda_4 \begin{cases} >0, & t \in [0, t') \\ =0 & , t \in [t', t'''] \\ <0, & t \in (t''', t_1] \end{cases}. \tag{57}$$

The costate system (49) also has analytical solutions. As:

$$\lambda_3(t'') = \lambda_4(t'') = \dot{\lambda}_4(t'') = 0, \quad v(t'') = 1 + t'', \quad \varphi(t'') = \pi / 4, \quad H(t'') = 0, \tag{58}$$

the determination can be also done of:

$$\lambda_1(t) = \lambda_2(t) = \frac{1}{(1 + t'')\sqrt{2}}. \tag{59}$$

so that the Kelley condition (54) is satisfied too. Optimal controls (55) also satisfy the necessary conditions [4, 6] for the optimality of junctions between singular and nonsingular subarcs of optimal trajectory. Fig. 4 shows the center of inertia trajectory and on the Fig.5 are corresponding diagrams .

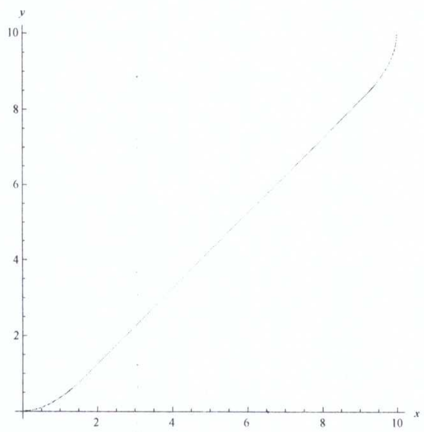


**Figure 4.** Center of inertia trajectory

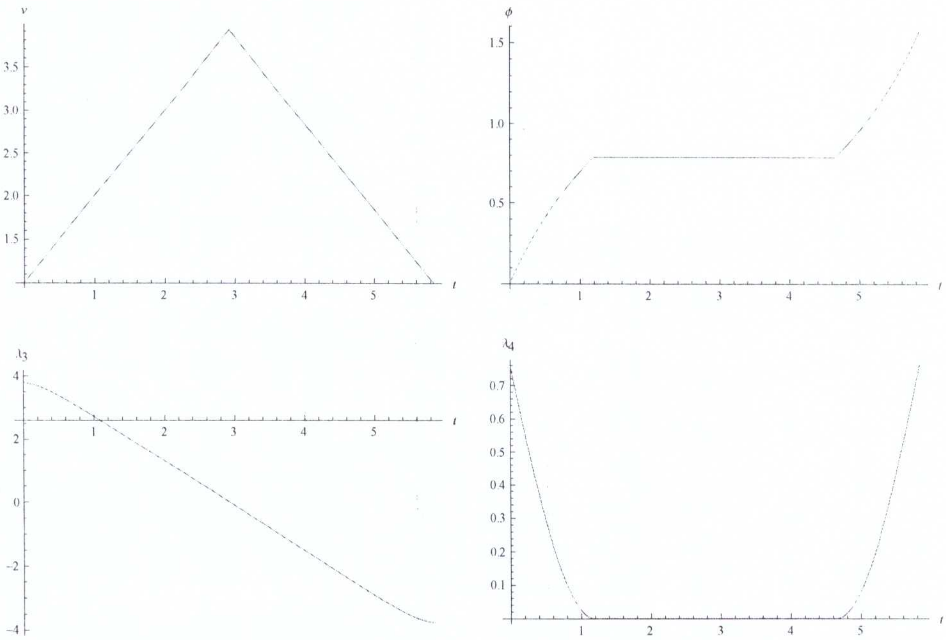


**Figure 5.** Diagrams $v(t)$, $\varphi(t), \lambda_3(t), \lambda_4(t)$

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