Mass minimization of an AFG Timoshenko beam with coupled axial and bending vibrations*

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ABSTRACT

The paper considers mass minimization of an axially functionally graded (AFG) Timoshenko beam of a variable cross-sectional area, with a specified fundamental frequency. The analyzed case of coupled axial and bending vibrations involves contour conditions as the cause of coupling. The problem is solved applying Pontryagin's maximum principle, where the beam cross-sectional area is taken for control. The two-point boundary value problem is obtained, and the shooting method is used to solving it. The property of self-adjoint systems is deployed. The percent saving of the beam mass is determined, achieved by using the beam of an optimum variable square cross-section as compared to the beam of a constant cross-section. The procedure developed by the author in his earlier papers is extended herein to the case of a limited cross-sectional area. The second generalization relates to the general case of contour conditions at the beam ends.

1. Introduction

Some elastic bodies can be modeled to obtain sufficient accuracy by using Euler-Bernoulli or Timoshenko beam [1] in the analysis of their vibrations. Determination of optimal shapes applying different optimization criteria is an engineering task of utter importance. In a static sense, these problems are associated with the optimization of columns against buckling. Papers [2,3] employ a buckling optimization approach based on the Hencky bar-chain model of beams, whereas [4-8] consider a series of shape optimization problems of columns against buckling by implementing Pontryagin's maximum principle.

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In this paper, the dynamic aspect of beam optimization includes references related to the problem of mass minimization of rods when the value of fundamental frequency of these rods oscillation is specified. Among the most outstanding references relevant to this problem are [9-11] where the minimum mass optimization problem is considered within the framework of axial vibrations of cantilever beams with a variable cross-section carrying concentrated mass at their free ends. Moreover, [9,10] deal with the tapered type of beams, whereas [11] is concerned with stepped ones. Paper [12] analyzes the impact of a variable cross- section (the linear relationship between the second area moment and the area) of Euler-Bernoulli beams in bending vibrations on the extremal values of the fundamental frequency, and [13] studies the conditions of equivalence between maximum frequency and minimum mass optimization problems. In [14] an optimality criterion method for weight minimization of structures under the fundamental frequency constraint in the form of inequality is presented. Mass minimization problems with multiple frequency constraints of equality and inequality types are reported in [15,16]. Paper [17], besides frequency constraints, studies simultaneously stress, displacement and cross-sectional area multiple constraints. The mass minimization of structures together with maximization of structural strain energy (multiobjective optimization problems) can be found in [18]. Paper [19] provides an example of the application of bio-inspired algorithms (firefly algorithm, bat algorithm and cuckoo search algorithm) in the problem of mass minimization of a single-girder bridge crane. Today, this type of algorithms has increasing application in complex optimization problems with multiple objective functions, design variables and equality and inequality constraints, for more details refer to [20].

Based on a review of available literature, it is evident that no results are reported in the field of mass minimization of beams with prescribed fundamental frequency, whose oscillatory behavior can be represented in the form of coupling between fundamental types of oscillations (axial vibrations, bending vibrations, torsion vibrations). The fact mentioned above makes this field of research potentially attractive. Otherwise, the coupling of different types of oscillations can be conditioned, for instance, by the cross-section geometry of a beam (cross-section with one symmetry axis, which causes coupling of bending and torsion vibrations [21]), or by complex contour conditions at the beam ends (coupling of bending and axial vibrations [22]).

Our paper [23] considers the second cause of coupled oscillations for the case of simply supported Euler Bernoulli beam with inclined right end. Moreover, the approach from [4-8] based on applying Pontryagin's maximum principle is extended to the mass minimization problem of the mentioned simply supported beam with coupled bending and axial vibrations at prescribed fundamental frequency. Here, it is worth highlighting that in [4-8] it was for the first time in literature that the occurrence of the self-adjoint system is observed in the problems of determining optimal shapes. This allows for twice fewer differential equations of the TPBVP problem of the Maximum principle as compared to classical problems and it is known that the difficulties in numerical solving are the main reason why the authors of the works dealing with the application of the mathematical theory of optimal processes often avoid using Pontryagin's principle.

Our later works [24-25] research an AFG Timoshenko cantilever beam instead of Euler Bernoulli homogeneous beams.

This paper extends additionally the procedure of shape optimization to the general case of contour conditions and introduces the limits of cross-sectional dimensions. It is necessary to limit the cross-sectional area from the bottom side so as not to disturb the strength of the beam. The upper limit can be defined by the beam initial shape that yields optimal shape by material removal, space limits, or to ensure the validity of the appropriate theory (Euler Bernoulli or, in this case, Timoshenko theory).

2. Problem statement for determining the optimal profile shape

The Timoshenko beam [1] (Fig.1) of length L, variable cross-sectional area $A(z)$ and axial moment of inertia $I_x(z) = sA(z)^2$, where coefficient s depends on the cross-sectional shape (for the square cross-section $= 1/12$, while in the case of circular cross-section $s = 1/4\pi$), is considered. In the case of AFG material, the density $\rho(z)$, Young's modulus of elasticity $E(z)$ and the shear modulus $G(z)$, are variable along the beam axis. At the right end a body of mass M_r and moment of inertia J_{Cr} is fixed eccentrically to the central axis, where the position of the center of mass is defined by quantities e_r and h_r. The corresponding stiffnesses of springs at the right end are c_r, c_{1r} and c_{2r}. All quantities given at the left end have index *l* instead of index r.

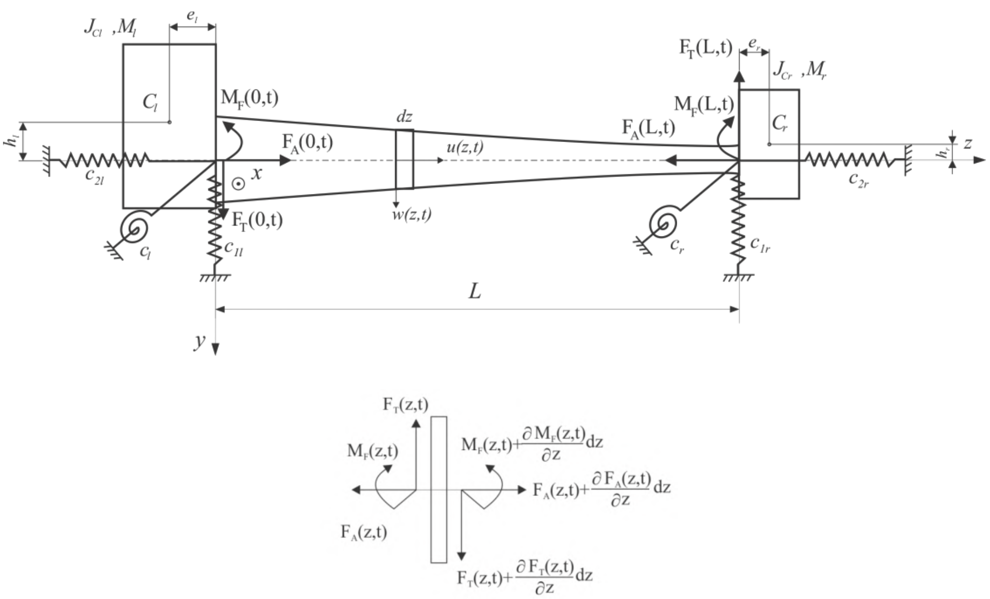

Fig. 1 AFG Timoshenko beam of variable cross-section with bodies eccentrically located at the beam ends

Differential equations of Timoshenko beams, oscillating in the axial and bending direction, in the case of the linear theory, can be derived based on dynamic equations of the elementary particle of the beam of mass $dm = \rho(z)A(z)dz$ and a corresponding moment of inertia of masses $dJ_x = \rho(z)I_x(z)dz$:

$$\begin{aligned}
&(\rho(z)A(z)dz)\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{\partial}{\partial z}\left[F_A(z,t)\right]dz,\\
&(\rho(z)A(z)dz)\frac{\partial^2 w(z,t)}{\partial t^2} = \frac{\partial}{\partial z}\left[F_T(z,t)\right]dz,\\
&(\rho(z)I_x(z)dz)\frac{\partial^2 \phi(z,t)}{\partial t^2} = \left[F_T(z,t)\right]dz - \frac{\partial}{\partial z}\left[M_F(z,t)\right]dz,
\end{aligned} \tag{1}$$

where $u(z,t)$ and $w(z,t)$ are axial and transverse displacements, $\phi(z,t)$ is the cross-sectional angle of rotation, $F_A(z,t)$ represents the axial force:

$$F_A(z,t) = E(z)A(z)\frac{\partial u(z,t)}{\partial z}, \tag{2}$$

the bending moment is given by the expression:

$$M_F(z,t) = -E(z)I_x(z)\frac{\partial \phi(z,t)}{\partial z}, \tag{3}$$

where for Timoshenko beams [1] the slope angle of the elastic line is:

$$\frac{\partial w(z,t)}{\partial z} = \phi(z,t) + \frac{F_T(z,t)}{kA(z)G(z)}, \tag{4}$$

where $F_T(z,t)$ is the transverse force and k is the Timoshenko coefficient.

The system of linear differential equations (1-4) is solved by the method of separation of variables [1]:

$$\begin{gathered} w(z,t) = W(z)T(t), \\ \phi(z,t) = \varphi(z)T(t), u(z,t) = U(z)T(t), F_A(z,t) = F_a(z)T(t), \\ F_T(z,t) = F_t(z)T(t), M_F(z,t) = M_f(z)T(t), \end{gathered} \tag{5}$$

where $\frac{\partial^2 T(t)}{\partial t^2} = -\omega^2 T(t)$ and ω represents a circular frequency. If we want all functions (5) to retain their physical dimensions and units, the function $T(t)$ will be considered to be dimensionless.

Further procedure yields the following differential equations:

$$\begin{gathered} \frac{\partial U(z)}{\partial z} = \frac{F_a(z)}{E(z)A(z)}, \frac{\partial W(z)}{\partial z} = \varphi(z) + \frac{F_t(z)}{kA(z)G(z)}, \frac{\partial \varphi(z)}{\partial z} = -\frac{M_f(z)}{E(z)sA(z)^2}, \\ \frac{\partial F_a(z)}{\partial z} = -\omega^2 \rho(z)A(z)U(z), \frac{\partial F_t(z)}{\partial z} = -\omega^2 \rho(z)A(z)W(z), \\ \frac{\partial M_f(z)}{\partial z} = F_t(z) + \omega^2 \rho(z)sA(z)^2 \varphi(z). \end{gathered} \tag{6}$$

Optimization problem, considered in this paper, includes defining the function of change of the cross-sectional area $A(z)$ that will lead to the Timoshenko beam mass minimization, where the fundamental frequency of oscillation $\omega_1 = \omega^*$ is specified. In that regard, the functional that is minimized is of the form:

$$J = \int_0^L \rho(z)A(z)dz, \tag{7}$$

differential equations (6) represent the equations of state.

Contour conditions at the ends can be written using differential equations of planar motion for each of the added rigid body in a way as described in more detail in [22]. Contour conditions at the left end are of the form:

$$\begin{gathered} M_l\omega^2(U(0) + h_l\varphi(0)) + F_a(0) - c_{2l}U(0) = 0, \\ M_l\omega^2(W(0) - e_l\varphi(0)) + F_t(0) - c_{1l}W(0) = 0, \\ J_{Cl}\omega^2\varphi(0) - M_f(0) - c_l\varphi(0) + e_l(F_t(0) \\ -c_{1l}W(0)) + h_l(-F_a(0) + c_{2l}U(0)) = 0, \end{gathered} \tag{8}$$

whereas at the right end:

$$\begin{aligned}
&M_r\omega^2(U(L)+h_r\varphi(L))-F_a\left(L\right)-c_{2r}U(L)=0,\\
&M_r\omega^2(W(L)+e_r\varphi(L))-F_t\left(L\right)-c_{1r}W(L)=0,\\
&J_{Cr}\omega^2\varphi(L)+M_f\left(L\right)-c_r\varphi(L)+e_r(F_t\left(L\right)\\
&+c_{1r}W(L))+h_r(F_a\left(L\right)+c_{2r}U(L))=0.
\end{aligned}\tag{9}$$

3. Determining the optimal profile shape of a beam by applying Pontryagin's maximum principle

The optimal control problem (6-9) will be solved by applying Pontryagin's maximum principle [26]. To this end, let us write Pontryagin's function:

$$\begin{aligned}
&H=\lambda_0\rho\left(z\right)A(z)+\lambda_U(z)\frac{F_a\left(z\right)}{E\left(z\right)A(z)}+\lambda_W(z)(\varphi\left(z\right)+\frac{F_t\left(z\right)}{k\,A(z)G(z)})\\
&+\lambda_\varphi(z)\frac{-M_f\left(z\right)}{E\left(z\right)sA(z)^2}-\lambda_{F_a}(z)\omega^2\rho\left(z\right)A(z)U\left(z\right)\\
&-\lambda_{F_t}(z)\omega^2\rho\left(z\right)A(z)W\left(z\right)+\lambda_{M_f}(z)(F_t\left(z\right)+\omega^2\rho\left(z\right)sA(z)^2)
\end{aligned}\tag{10}$$

where $\lambda_0,\lambda_U(z),\lambda_W(z),\lambda_\varphi(z),\lambda_{F_a}(z),\lambda_{F_t}(z),\lambda_{M_f}(z)$ are costate variables, which satisfy the coupled system of equations:

$$\begin{aligned}
&\frac{\partial\lambda_U(z)}{\partial z}=\lambda_{F_a}(z)\omega^2\rho\left(z\right)A(z),\ \frac{\partial\lambda_W(z)}{\partial z}=\lambda_{F_t}(z)\omega^2\rho\left(z\right)A(z),\\
&\frac{\partial\lambda_\varphi(z)}{\partial z}=-\lambda_W(z),\\
&\frac{\partial\lambda_{F_a}(z)}{\partial z}=\frac{-\lambda_U(z)}{E(z)A(z)},\ \frac{\partial\lambda_{F_t}(z)}{\partial z}=\frac{-\lambda_W(z)}{kA(z)G(z)}-\lambda_{M_f}(z),\\
&\frac{\partial\lambda_{M_f}(z)}{\partial z}=\frac{\lambda_\varphi(z)}{E(z)sA(z)^2},
\end{aligned}\tag{11}$$

where λ_0 is an arbitrary non-positive constant. In problems of this type it is commonly taken that $\lambda_0=-1$.

The transversality conditions [26] can be represented in the form as follows:

$$\begin{aligned}
&(\lambda_U(z)\Delta U(z)+\lambda_W(z)\Delta W(z)+\lambda_\varphi(z)\Delta\varphi(z)+\lambda_{F_a}(z)\Delta F_a(z)+\\
&\lambda_{F_t}(z)\Delta F_t(z)+\lambda_{M_f}(z)\Delta M_f(z))\ \Big|_0^L=0,
\end{aligned}\tag{12}$$

where $\Delta(\cdot)$ is an asynchronous variation. Based on initial and final conditions (8,9), the following variation dependencies are obtained at the left end:

$$(M_l\omega^2 - c_{2l})\Delta U(0) + M_l\omega^2 h_l \Delta\varphi(0) + \Delta F_a(0) = 0,$$
$$(M_l\omega^2 - c_{1l})\Delta W(0) - M_l\omega^2 e_l \Delta\varphi(0) + \Delta F_t(0) = 0,$$
$$(J_{Cl}\omega^2 - c_l)\Delta\varphi(0) - M_f(0) + e_l\Delta F_t(0) - c_{1l}e_l\Delta W(0)$$
$$-h_l\Delta F_a(0) + c_{2l}h_l\Delta U(0) = 0, \tag{13}$$

and at the right end:

$$(M_r\omega^2 - c_{2r})\Delta U(L) + M_r\omega^2 h_r \Delta\varphi(L) - \Delta F_a(L) = 0,$$
$$(M_r\omega^2 - c_{1r})\Delta W(L) + M_r\omega^2 e_r \Delta\varphi(L) - \Delta F_t(L) = 0,$$
$$(J_{Cr}\omega^2 - c_r)\Delta\varphi(L) + \Delta M_f(L) + e_r\Delta F_t(L) + e_r c_{1r}\Delta W(L)$$
$$+h_r\Delta F_a(L) + h_r c_{2r}\Delta U(L) = 0. \tag{14}$$

Substituting (13) and (14) in (12), the transversality conditions are obtained:

$$-(M_l\omega^2 - c_{2l})\lambda_{F_a}(0) + M_l\omega^2 h_l\lambda_{M_f}(0) + \lambda_U(0) = 0,$$
$$-(M_l\omega^2 - c_{1l})\lambda_{F_t}(0) - M_l\omega^2 e_l\lambda_{M_f}(0) + \lambda_W(0) = 0,$$
$$(J_{Cl}\omega^2 - c_l)\lambda_{M_f}(0) + \lambda_\varphi(0) + e_l\lambda_W(0) + c_{1l}e_l\lambda_{F_t}(0)$$
$$-h_l\lambda_U(0) - c_{2l}h_l\lambda_{F_a}(0) = 0,$$
$$-(M_r\omega^2 - c_{2r})\lambda_{F_a}(L) + M_r\omega^2 h_r\lambda_{M_f}(L) - \lambda_U(L) = 0,$$
$$-(M_r\omega^2 - c_{1r})\lambda_{F_t}(L) + M_r\omega^2 e_r\lambda_{M_f}(L) - \lambda_W(L) = 0,$$
$$(J_{Cr}\omega^2 - c_r)\lambda_{M_f}(L) - \lambda_\varphi(L) + e_r\lambda_W(L) - e_r c_{1r}\lambda_{F_t}(L)$$
$$+h_r\lambda_U(L) - h_r c_{2r}\lambda_{F_a}(L) = 0.$$

(15)

If the conjugate vector coordinates are expressed via state quantities using the scalar parameter p:

$$\lambda_U = pF_a,\ \lambda_W = pF_t, \lambda_\varphi = -pM_f,\ \lambda_{M_f} = p\varphi,\ \lambda_{F_t} = -pW,\ \lambda_{F_a} = -pU, \tag{16}$$

it can be noted that differential equations of the coupled system (11) are reduced to the governing system (6), and that the transversality conditions (15) are satisfied in the case when the conditions at the left end (8) and at the right end (9) are satisfied. This has been noted in the shape optimization problems reported by Atanacković et al. [4-8] and has significantly facilitated the application of Pontryagin's maximum principle. It is very well known that numerical difficulties related to computations of the costate variables are those that are limiting the application of maximum principle.

Optimal controls $A(z)$ are defined from the maximum condition of Pontryagin's function (10):

$$\frac{\partial H}{\partial A(z)}=0,\qquad \frac{\partial^2 H}{\partial A(z)^2}<0, \tag{17}$$

which, considering (16), is reduced to the conditions:

$$-\frac{pF_a(z)^2}{E(z)A(z)^2}-\frac{pF_t(z)^2}{kG(z)A(z)^2}-\frac{2pM_f(z)^2}{sA(z)^3E(z)}+$$
$$+2ps\omega^2A(z)\varphi(z)^2\rho(z)+(-1+p\omega^2U(z)^2+p\omega^2W(z)^2)\rho(z)=0, \tag{18}$$

$$2p\left(\frac{F_a(z)^2}{E(z)A(z)^3}+\frac{F_t(z)^2}{kG(z)A(z)^3}+\frac{3M_f(z)^2}{sA(z)^4E(z)}+s\omega^2A(z)\varphi(z)^2\rho(z)\right)<0. \tag{19}$$

Based on (19), without loss of generality, it can be taken that $p=-1\frac{s^2}{m^2}$, so that the expression for defining optimal control is reduced to the 4th degree polynomial with respect to $A(z)$:

$$\left(\frac{F_a(z)^2}{E(z)}+\frac{F_t(z)^2}{kG(z)}\right)A(z)+\frac{2M_f(z)^2}{sE(z)}-$$
$$-2s\omega^2A(z)^4\varphi(z)^2\rho(z)-(1+\omega^2U(z)^2+\omega^2W(z)^2)\rho(z)A(z)^3=0. \tag{20}$$

The procedure of numerical solving of the two-point boundary value problem (6-9,20) consists of the three-parameter shooting that involves selecting $F_a(0), F_t(0), M_f(0)$ (where $U(0), W(0), \varphi(0)$ can be calculated from (8)) to satisfy the relations (9). If numerical solving is performed in the program package *WolframMathematica* [27] using function NDSolve[…], it is not necessary to express *A(z)* from (20) in analytical form via state quantity, because this function contains in itself the procedure for numerical solving of the system of differential and ordinary equations.

In the case of restrictions imposed on the cross-sectional area:

$$A_{\min}\le A(z)\le A_{\max} \tag{21}$$

it is necessary to check whether the values obtained from (20) satisfy the restrictions (21). If they are lower or higher compared to the permissible limit values, the cross-sectional areas are constant over those intervals, amounting to $A_{\max}$ or $A_{\min}$. As a rule, when solving such problems, the optimal shape is first determined without considering the restrictions (21). Thereafter, it is attempted to find a control that satisfies all conditions of the Maximum principle of such structure that on the segments where an area larger than $A_{\max}$ is obtained by solving (20), it is taken that in that segment a constant cross-section is of maximum permissible area. Similarly, in the case when the area is smaller, but it is then taken $A_{\min}$.

Positions z_i where a variable cross-section joins a constant cross-section of the maximum area are determined from the conditions:

$$\left(\frac{F_a(z_i)^2}{E(z_i)}+\frac{F_t(z_i)^2}{kG(z_i)}\right)A_{\max}+\frac{2M_f(z_i)^2}{sE(z_i)} \tag{22}$$
$$-2s\omega^2 A_{\max}{}^4\varphi(z_i)^2\rho(z_i)-(1+\omega^2U(z_i)^2+\omega^2W(z_i)^2)\rho(z_i)A_{\max}{}^3=0,$$

however, in the case of merging with a segment of the constant cross-section of the minimum area:

$$\left(\frac{F_a(z_i)^2}{E(z_i)}+\frac{F_t(z_i)^2}{kG(z_i)}\right)A_{\min}+\frac{2M_f(z_i)^2}{sE(z_i)}- \tag{23}$$
$$-2s\omega^2 A_{\min}{}^4\varphi(z_i)^2\rho(z_i)-(1+\omega^2U(z_i)^2+\omega^2W(z_i)^2)\rho(z_i)A_{\min}{}^3=0.$$

4. Numerical example

The shape optimization procedure will be presented using the example of a cantilever beam of a square cross-section, length L=1m, with a rigid body placed eccentrically at the free end, as shown in Fig. 2. The rigid body has mass $M_r=10\,kg$ and moment of inertia $J_{Cr}=2.5\,kg\,m^2$. Axial and transverse eccentricities of the rigid body amount to $e_r=e=h_r=h=0.5m$. In AFG material considered herein the laws of change in density and modulus of elasticity are taken as in [24,25]:

$$\rho(z)=\rho_0(1-0.8\cos(\pi z)),\ \rho_0=7850\frac{kg}{m^3},$$
$$E(z)=E_0(1-0.2\cos(\pi z)),\ E_0=2.068\cdot 10^{11}\frac{N}{m^2}. \tag{24}$$

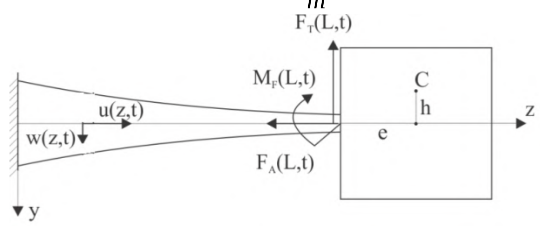

Fig. 2 [25] Cantilever beam of a variable square cross-section

For Timoshenko beams of a square cross-section, $= 1/12$. The Timoshenko coefficient, in this case, amounts approximately to $k=\frac{5}{6}$. The shear modulus is

defined using the Poisson coefficient ν from the expression $G(z) = \frac{E(z)}{2(1+\nu)}$, where for its value it is taken here that $\nu = 0.3$.
Also, let the required value of the fundamental frequency be $f = 10\ Hz$ which leads to the fundamental circular frequency $\omega^* = 20\pi\ Hz$.
Let us seek a solution of the optimization problem first for the case when there are not restrictions to the cross-sectional area. When performing a three-parameter shooting in the program package *WolframMathematica* [27], three missing values of the three parameters at the left end are obtained:

$$F_a(0) = -219.631N, \quad F_t(0) = -451.863N, \quad M_f(0) = 848.826\,Nm, \qquad (25)$$

It should be noted here that from the contour conditions (8), when stiffnesses are of infinitely large values (in the case of a clamped left end), zero values of corresponding displacements U(0), W(0), φ(0) formally follow too.
Fig. 3 shows values of the optimum cross-sectional area shape and its corresponding sides of a square (red line). The dashed line denotes values corresponding to a constant cross-sectional area A^* and the side of a square a^*, respectively $A^* = 0.00207910\,m^2$, $a^* = 0.0455971m$ for which the fundamental circular frequency is $\omega^* = 20\pi\,Hz$, and which were obtained in paper [24].

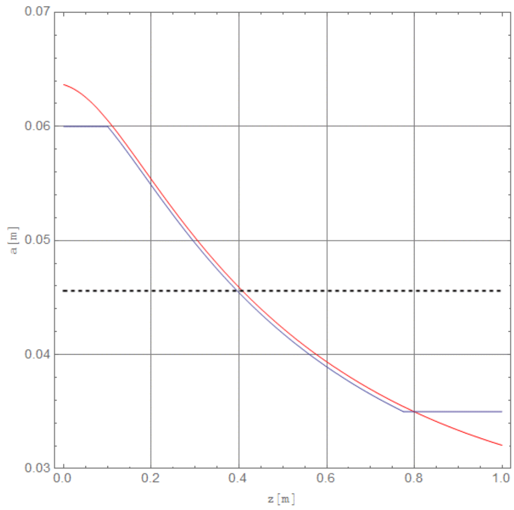

Fig. 3 Optimum cross-sectional side of a square *a(z)*

Relative material saving compared to the cantilever beam of a constant cross-section corresponding to the same circular frequency amounts to:

$$\Delta = \frac{\left(\int_0^L \rho(z)\, A_1 \mathrm{d}z - \int_0^L \rho(z) A(z) \mathrm{d}z\right)}{\int_0^L \rho(z)\, A_1 \mathrm{d}z} \mathrm{x}100\% = 23.38\%, \tag{26}$$

where numerical integration was done in (26).

Consider the case when the cross-sectional area is limited so that a_{max}=0.06*m*, a_{min}=0.035*m*. Therefore, it can be assumed that at the initial segment the beam is of the maximum possible area and at the end segment it is of the minimum area.

Besides unknow quantities $F_a(0)$, $F_t(0)$, $M_f(0)$, it is also necessary to determine positions $z_{1,2}$ of coupling occurrence between maximum of minimum corss-section varying in area.

These five parameters are chosen so that after numerical integration of the system (6) final conditions (9) as well as conditions (22,23) are satisfied at the coupling points. The values obtained as solutions are

$$\begin{gathered} F_a(0) = -208.038\,N, \quad F_t(0) = -446.939\,N, \quad M_f(0) = 826.645\,Nm \\ z_1 = 0.10013\,m,\; z_2 = 0.77526\,m \end{gathered} \tag{27}$$

and optimal shape is shown in Fig. 3 (blue line). In this case, the relative percent saving of the mass (26) is slightly lower and amounts to 22.97%.

5. Conclusions

This paper demonstrates the performance of shape optimization of AFG Timoshenko beam of a square cross-section with coupled axial and bending vibrations, where the beam mass minimization is done at specified fundamental frequency. In solving this optimization problem Pontryagin's maximum principle is applied. So far, Pontryagin’s maximum principle has been practically used for solving optimization problems in buckling so that in this paper its application is extended to optimization problems in oscillating body. The above procedure can be also applied to the general case of a cross-section such as circular, etc. The above procedure can be also applied to the general case of contour conditions at the beam ends, including bodies eccentrically positioned at both ends, different types of supports at beam ends, as well as clamping of the bodies with different springs. By taking infinitely large stiffnesses of appropriate springs, the model considered can be also extended to the cases when the corresponding displacements in the supports equal zero.

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