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Some critical remarks on "Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations"

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Abstract. In this manuscript, we generalize, improve, and enrich recent results established by Budhia et al. [L. Budhia, H. Aydi, A.H. Ansari, D. Gopal, Some new fixed point results in rectangular metric spaces with application to fractional-order functional differential equations, *Nonlinear Anal. Model. Control*, 25(4):580–597, 2020]. This paper aims to provide much simpler and shorter proofs of some results in rectangular metric spaces. According to one of our recent lemmas, we show that the given contractive condition yields Cauchyness of the corresponding Picard sequence. The obtained results improve well-known comparable results in the literature. Using our new approach, we prove that a Picard sequence is Cauchy in the framework of rectangular metric spaces. Our obtained results complement and enrich several methods in the existing state-of-art. Endorsing the materiality of the presented results, we also propound an application to dynamic programming associated with the multistage process.

Keywords: rectangular metric space, triangular α-admissible, α-regular with respect to η, dynamic programing, fixed point.

1 Introduction and preliminaries

It is well known that the Banach contraction principle [5] is one of the most essential and attractive results in nonlinear analysis and mathematical analysis in general. The whole fixed point theory is a significant subject in different fields as geometry, differential equations, informatics, physics, economics, engineering, and many others (see [8, 23, 25, 27]). After the solutions are guaranteed, the numerical methodology has been adopted to obtain the approximated solution [28].

In 2000, generalized metric spaces were introduced by Branciari [6] in such a way that the triangle inequality is replaced by the quadrilateral inequality $d(x,y) \leqslant d(x,z) + d(z,u) + d(u,y)$ for all pairwise distinct points x, y, z, and u. Any metric space is a generalized metric space, but in general, generalized metric space might not be a metric space. Various fixed point results were established in such spaces (see [3,4,7,9–13,16,26] and references therein).

In this paper, we will discuss some results recently established in [7]. Firstly, we propound some basic notions and definitions, which are necessary for the subsequent analysis.

Definition 1. Let M be a nonempty set, and let $d : M \times M \to [0,+\infty)$ satisfy the following conditions: for all $p, r \in M$ and all distinct $u, v \in M$ each of them different from p and r,

(i) $d(p,r) = 0$ if and only if $p = r$;
(ii) $d(p,r) = d(r,p)$;
(iii) $d(p,r) \leqslant d(p,u) + d(u,v) + d(v,r)$ (quadrilateral inequality).

Then the function d is called a rectangular metric, and the pair (M,d) is called a rectangular metric space (in short RMS).

Notice that the definitions of convergence and the Cauchyness of sequences in rectangular metric spaces are similar to those found in the standard metric spaces. Also, a rectangular metric space (M,d) is complete if every Cauchy sequence in M is convergent.

Samet et al. [24] introduced the concept of an $\alpha{-}\psi$-contractive mapping and proved fixed point theorems for such mappings. Karapınar [13] extended the concepts given in [24] to obtain the existence and uniqueness of a fixed point of $\alpha{-}\psi$-contraction mappings in the setting of rectangular metric spaces. After that, Salimi et al. [23] introduced a modified $\alpha{-}\psi$-contractive mapping and obtained some fixed point theorems in the complete metric spaces. Alsulami et al. [1] established some fixed point theorems for an $\alpha{-}\psi$-rational-type contractive mapping in the context of rectangular metric spaces.

Let Ψ be the family of all functions $\psi : [0,+\infty) \to [0,+\infty)$ such that ψ is nondecreasing and continuous (so-called an altering distance function) and $\psi(t) = 0$ if and only if $t = 0$ (for more details, see [15,28]).

Definition 2. (See [23].) Let T be a self-mapping on a metric space (M,d), and let $\alpha, \eta : M \times M \to [0,+\infty)$ be two functions. Then T is called an α-admissible mapping with respect to η if $\alpha(p,r) \geqslant \eta(p,r)$ implies that $\alpha(Tp,Tr) \geqslant \eta(Tp,Tr)$ for all $p, r \in M$.

If $\eta(p,r) = 1$ for all $p, r \in M$, then T is called an α-admissible mapping.

T is said to be a triangular α-admissible mapping if for all $p, r, q \in M$, the following holds: $(\alpha(p,r) \geqslant 1$ and $\alpha(r,q) \geqslant 1)$ implies $\alpha(p,q) \geqslant 1$.

Otherwise, a rectangular metric space (M,d) is said to be an α-regular with respect to η if for any sequence p_n in M such that $\alpha(p_n, p_{n+1}) \geqslant \eta(p_n, p_{n+1})$ for all $n \in \mathbb{N}$ and $p_n \to p$ as $n \to +\infty$, implies $\alpha(p_n, p) \geqslant \eta(p_n, p)$.

For more details on the triangular α-admissible mapping, see [14, pp. 1, 2]. In this paper the following results play an important role.

Lemma 1. (*See* [14, Lemma 7]). *Let T be a triangular α-admissible mapping. Assume that there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geqslant 1$. Define a sequence $\{p_n\}$ by $p_n = T^n p_0$. Then*

$$\alpha(p_m, p_n) \geqslant 1 \quad \textit{for all } m, n \in \mathbb{N} \cup \{0\} \textit{ with } m < n.$$

The following definition is due to [2], where the class of C-functions is introduced.

Definition 3. A C-function $F : [0, +\infty)^2 \to \mathbb{R}$ is a continuous function such that for all $x, y \in [0, +\infty)$:

(i) $F(x, y) \leqslant x$;
(ii) $F(x, y) = x$ implies that either $x = 0$ or $y = 0$.

The letter $\mathcal{C}$ will denote the class of all C-functions. For detailed description and examples of C-functions, we refer the reader to [2, 7].

The following remark plays a significant role in the rest of this article.

Remark 1. It is worth to mention that for each C-function, $F(x, y) < x$ if $x \neq 0$ and $y \neq 0$.

In [7] the authors proved the following two theorems.

Theorem 1. *Let (M, d) be a complete Hausdorff rectangular metric space, and let $T : M \to M$ be an α-admissible mapping with respect to η. Suppose there exist $F \in C$ and $\psi, \phi \in \Psi$ such that, for $p, r \in M$,*

$$\alpha(p, r) \geqslant \eta(p, r) \quad \textit{implies} \quad \psi\bigl(d(Tp, Tr)\bigr) \leqslant F\bigl(\psi\bigl(m(p, r)\bigr), \phi\bigl(m(p, r)\bigr)\bigr),$$

where

$$m(p, r) = \max\left\{d(p, r), d(p, Tp), d(r, Tr), \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)}, \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}\right\}.$$

Assume that:

(i) *there exists $u_0 \in M$ for which $\alpha(u_0, Tu_0) \geqslant \eta(u_0 Tu_0)$;*
(ii) *for all $u, v, w \in M$, $(\alpha(u, v) \geqslant \eta(u, v)$ and $\alpha(v, w) \geqslant \eta(v, w))$ implies $\alpha(u, w) \geqslant \eta(u, w)$;*
(iii) *T is continuous or M is α-regular with respect to η.*

Then there exists $x \in M$ such that $T^n x = x$ for some $n \in \mathbb{N}$, i.e., x is a periodic point. If in addition, $\alpha(x, Tx) \geqslant \eta(x, Tx)$ for each periodic point x, then T has a fixed point.

Theorem 2. *To ensure the uniqueness of the fixed point in Theorem* 1*, the authors add the following condition*:

$$\alpha(a, b) \geqslant \eta(a, b) \quad \textit{for all } a, b \in F(T) = \{w \in M\colon Tw = w\}.$$

Taking $F(x, y) = x - y$, the authors obtained the following corollaries.

Corollary 1. *Let* (M,d) *be a Hausdorff and complete rectangular metric space, and let* $T: M \to M$ *be an* α*-admissible mapping with respect to* η*. Assume that there exists* $\psi, \phi \in \Psi$ *such that, for* $p, r \in M$*,*

$$\alpha(p,r) \geqslant \eta(p,r) \quad \textit{implies} \quad d(Tp,Tr) \leqslant \psi\big(m(p,r)\big) - \phi\big(m(p,r)\big),$$

where

$$m(p,r) = \max\left\{d(p,r), d(p,Tp), d(r,Tr), \frac{d(p,Tp)d(r,Tr)}{1+d(p,r)}, \frac{d(p,Tp)d(r,Tr)}{1+d(Tp,Tr)}\right\}.$$

Also suppose that the following assertions are contended:

(i) *there exists* $u_0 \in M$ *for which* $\alpha(u_0, Tu_0) \geqslant \eta(u_0 Tu_0)$;
(ii) *for all* $u, v, w \in M$, $(\alpha(u,v) \geqslant \eta(u,v)$*and* $\alpha(v,w) \geqslant \eta(v,w))$ *implies* $\alpha(u,w) \geqslant \eta(u,w)$;
(iii) T *is continuous or* M *is* α*-regular with respect to* η*.*

Then T *has a periodic point* $x \in M$*. If in addition,* $\alpha(x,Tx) \geqslant \eta(x,Tx)$ *holds for each periodic point, then* T *has a fixed point. Moreover, if for all* $a, b \in F(T)$*, we have* $\alpha(a,b) \geqslant \eta(a,b)$*, then the fixed point is unique.*

Taking $\psi(t) = t$ in Corollary 1, the authors obtained the following result.

Corollary 2. *Let* (M,d) *be a Hausdorff and complete rectangular metric space. Let* $T: M \to M$ *be an* α*-admissible mapping with respect to* η*. Assume that there exists* $\phi \in \Psi$ *such that, for* $x, y \in M$*,*

$$\alpha(p,r) \geqslant \eta(p,r) \quad \textit{implies} \quad d(Tp,Tr) \leqslant m(p,r) - \phi\big(m(p,r)\big),$$

where

$$m(p,r) = \max\left\{d(p,r), d(p,Tp), d(r,Tr), \frac{d(p,Tp)d(r,Tr)}{1+d(p,r)}, \frac{d(p,Tp)d(r,Tr)}{1+d(Tp,Tr)}\right\}.$$

Also suppose that the following assertions hold:

(i) *there exists* $u_0 \in M$ *for which* $\alpha(u_0, Tu_0) \geqslant \eta(u_0 Tu_0)$;
(ii) *for all* $u, v, w \in M$, $(\alpha(u,v) \geqslant \eta(u,v)$*and* $\alpha(v,w) \geqslant \eta(v,w))$ *implies* $\alpha(u,w) \geqslant \eta(u,w)$;
(iii) T *is continuous or* M *is* α*-regular with respect to* η*.*

Then T *has a periodic point* $x \in M$*. If in addition,* $\alpha(x,Tx) \geqslant \eta(x,Tx)$ *for each periodic point, then* T *has a fixed point. Moreover, if for all* $a, b \in F(T)$*, we have* $\alpha(a,b) \geqslant \eta(a,b)$*, then the fixed point is unique.*

Consider $\phi(t) = (1-q)t$ for $0 < q < 1$ in Corollary 2.

Corollary 3. *Let* (M,d) *be a Hausdorff and complete rectangular metric space. Let* $T: M \to M$ *be an* α*-admissible mapping with respect to* η *such that, for* $p, r \in M$*,*

$$\alpha(p,r) \geqslant \eta(p,r) \quad \textit{implies} \quad d(Tp,Tr) \leqslant qm(p,r),$$

where $m(p,r)$ *is the same as in Corollary* 2*. Suppose also that the following hold*:

(i) *there exists* $u_0 \in M$ *for which* $\alpha(u_0, Tu_0) \geqslant \eta(u_0 Tu_0)$;
(ii) *for all* $u, v, w \in M$, $(\alpha(u,v) \geqslant \eta(u,v)$*and* $\alpha(v,w) \geqslant \eta(v,w))$ *implies* $\alpha(u,w) \geqslant \eta(u,w)$;
(iii) T *is continuous or* M *is* α*-regular with respect to* η.

Then T *has a periodic point* $x \in M$. *If in addition,* $\alpha(x, Tx) \geqslant \eta(x, Tx)$ *for each periodic point, then* T *has a fixed point. Moreover, if for all* $a, b \in F(T)$, *we have* $\alpha(a,b) \geqslant \eta(a,b)$, *then the fixed point is unique.*

In the sequel the authors in [7] gave two examples, which support their obtained theoretical results. In the next example, rectangular metric space (M,d) is not Hausdorff, and the mapping T has no fixed point. So the hypothesis that (M,d) is Hausdorff does not guarantee the existence of a fixed point.

Example 1. Let $M_1 = \{0,2\}$, $M_2 = \{1, 1/2, 1/3, \dots\}$, and $M = M_1 \cup M_2$. Define $d : M \times M \to [0,+\infty)$ as follows:

$$d(p,r) = \begin{cases} 0, & p = r \\ 1, & p \neq r \text{ and } \{p,r\} \subset M_1 \text{ or } \{p,r\} \subset M_2, \\ r, & p \in M_1 \text{ and } r \in M_2, \\ p, & p \in M_2 \text{ and } r \in M_1. \end{cases}$$

Then (M,d) is a complete rectangular metric space. Note that (M,d) is not Hausdorff because there exists no $s > 0$ such that $B(0,s) \cap B(2,s) = \emptyset$. Given $\alpha, \eta : M \times M \to [0,+\infty)$ as

$$\alpha(p,r) = \begin{cases} 4, & p \neq 0 \text{ or } r \neq \frac{1}{j}, \\ 2, & p = 0 \text{ and } r = \frac{1}{j}, \end{cases} \qquad \eta(p,r) = 3, (p,r) \in M \times M.$$

Define $T : M \to M$ by

$$T(0) = \frac{1}{2}, \quad T(2) = 0, \quad \text{and} \quad T\left(\frac{1}{j}\right) = 0 \quad \text{for } \frac{1}{j} \in M_2.$$

For their convenience, the authors in [7] use the following symbols:

$$A_1 = d(p,r), \qquad A_2 = d(p,Tp), \qquad A_3 = d(r,Tr),$$
$$A_4 = \frac{d(p,Tp)}{1 + d(p,r)} \quad \text{and} \quad A_5 = \frac{d(p,Tp)}{1 + d(Tp,Tr)}.$$

Define the functions $F : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ and $\psi, \phi : [0,+\infty) \to [0,+\infty)$ as $F(s,t) = 5s/6$, $\psi(t) = 2t/3$, and $\phi(t) = t/4$, respectively. Using the obtained table, the authors easily checked that the following condition is valid.

$$\psi\big(d(Tp,Tr)\big) \leqslant F\big(\psi\big(m(p,r)\big), \phi\big(m(p,r)\big)\big)$$

whenever $\alpha(p,r) \geqslant \eta(p,r)$, but $F(T)$ is empty.

However, the given example is not correct, namely, it does not satisfy all the conditions of Theorem 2 [7, Thm. 1]. It is easy to check that T is not α-admissible with respect to η. Indeed,

$$\alpha\left(\frac{1}{j},0\right) = 4 \geqslant 3 = \eta\left(\frac{1}{j},0\right),$$

while

$$\alpha\left(T\frac{1}{j},T0\right) = \alpha\left(0,\frac{1}{2}\right) = 2 < 3 = \eta\left(T\frac{1}{j},T0\right).$$

Further, T is not continuous. Indeed, $d(1/j,0) = 1/j \to 0$ as $j \to +\infty$, but $d(T/j,T0) = d(0,1/2) = 1/2 \nrightarrow 0$ as $j \to +\infty$.

Also, $\alpha(0,0)$ is not defined, we do not know whether (M,d) is α-regular.

The following two lemmas, in the setting of rectangular metric spaces, are modifications of a well-known result in metric spaces (see, e.g., [22, Lemma 2.1]). Many known proofs of fixed point results in rectangular metric spaces become much simpler and shorter using both these lemmas.

Lemma 2. (*See* [12].) *Let (M,d) be a rectangular metric space, and let $\{p_n\}$ be a sequence in it with distinct elements ($p_n \neq p_m$ for $n \neq m$). Suppose that $d(p_n,p_{n+1})$ and $d(p_n,p_{n+2})$ tend to 0 as $n \to +\infty$ and that $\{p_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$, and the following sequences tend to ε^+ as $k \to +\infty$:*

$$\begin{gathered}\left\{d(p_{n(k)},p_{m(k)})\right\}, \quad \left\{d(p_{n(k)+1},p_{m(k)})\right\}, \quad \left\{d(p_{n(k)},p_{m(k)-1})\right\},\\ \left\{d(p_{n(k)+1},p_{m(k)-1})\right\}, \quad \left\{d(p_{n(k)+1},p_{m(k)+1})\right\}.\end{gathered}$$

Lemma 3. *Let $\{p_{n+1}\}_{n\in\mathbb{N}\cup\{0\}} = \{Tp_n\}_{n\in\mathbb{N}\cup\{0\}} = \{T^n p_0\}_{n\in\mathbb{N}\cup\{0\}}$, $T^0 p_0 = p_0$ be a Picard sequence in rectangular metric space (M,d) inducing by the mapping $T: M \to M$ and initial point $p_0 \in X$. If $d(p_n,p_{n+1}) < d(p_{n-1},p_n)$ for all $n \in \mathbb{N}$, then $p_n \neq p_m$ whenever $n \neq m$.*

Proof. Let $p_n = p_m$ for some $n,m \in \mathbb{N}$ with $n < m$, then $p_{n+1} = Tp_n = Tp_m = p_{m+1}$, and we acquire

$$d(p_n,p_{n+1}) = d(p_m,p_{m+1}) < d(p_{m-1},p_m) < \cdots < d(p_n,p_{n+1}),$$

which is a contradiction. □

In the proof of our results, the following exciting and significant proposition is used in the context of rectangular metric spaces.

Proposition 1. (*See* [17, *Prop.* 3].) *Suppose that $\{p_n\}$ is a Cauchy sequence in a rectangular metric space (M,d), and suppose $\lim_{n\to+\infty} d(p_n,p) = 0$. Then for all $r \in M$, $\lim_{n\to+\infty} d(p_n,r) = d(p,r)$. In particular, $\{p_n\}$ does not converge to r if $r \neq p$.*

2 Some improved results

In this section, we generalize and improve Theorem 1 along with its corollaries. Obtained generalizations extend the result in several directions. It may be noted: we use only one function $\alpha : M \times M \to [0, +\infty)$ instead of two α and η as used in [8, Defs. 2.3, 3.1.]. This is possible according to the results enunciated in [21, p. 2].

Note that we neither assume that the rectangular metric space (M, d) is Hausdorff, nor that the mapping d is continuous.

The authors [1, p. 6, line 6+)] claimed that the sequence $\{p_n\}$ in rectangular metric space (M, d) is a Cauchy sequence if $\lim_{n\to+\infty} d(p_n, p_{n+k}) = 0$ for all $k \in \mathbb{N}$. However, it is ambivalent. We rectify the proof that the sequence $\{p_n\}$ is Cauchy. For more details, we refer the reader to the noteworthy and informative article [20, p. 7].

Our first new result in this paper is the following.

Theorem 3. *Let (M, d) be a complete rectangular metric space, and let $T : M \to M$ be a triangular α-admissible mapping, where $\alpha : M \times M \to [0, +\infty)$. Assume there exist $F \in \mathcal{C}$ and $\psi, \phi \in \Psi$ such that, for $p, r \in M$,*

$$\alpha(p, r) \geqslant 1 \quad \textit{implies} \quad \psi\bigl(d(Tp, Tr)\bigr) \leqslant F\bigl(\psi\bigl(m(p, r)\bigr), \phi\bigl(m(p, r)\bigr)\bigr), \tag{1}$$

where

$$m(p, r) = \max\left\{d(p, r), d(p, Tp), d(r, Tr), \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)}, \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}\right\}.$$

Also, suppose that the following assertions are satisfied:

(i) *there exists $u_0 \in M$ such that $\alpha(u_0, Tu_0) \geqslant 1$;*
(ii) *T is continuous or (M, d) α-regular.*

Then T has a fixed point. Moreover, if for all $p, r \in F(T)$, we have $\alpha(p, r) \geqslant 1$, then the fixed point is unique.

Proof. Given $u_0 \in M$ such that

$$\alpha(u_0, Tu_0) \geqslant 1. \tag{2}$$

Define a sequence $\{u_n\}$ in X by $u_n = Tu_{n-1} = T^n u_0$ for all $n \in \mathbb{N}$. If $u_{k+1} = u_k$ for some $k \in \mathbb{N}$, then $Tu_k = u_k$, i.e., u_k is a fixed point of T, and the proof is completed in this case.

From now, suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using (2) and the fact that T is an α-admissible mapping, we have

$$\alpha(u_1, u_2) = \alpha(Tu_0, Tu_1) \geqslant 1.$$

By induction we get

$$\alpha(u_n, u_{n+1}) \geqslant 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

In the first step, we will show that the sequence $\{d(u_n, u_{n+1})\}$ is nonincreasing and $d(u_n, u_{n+1}) \to 0$ as $n \to +\infty$. From (1) we have

$$\psi\bigl(d(u_n, u_{n+1})\bigr) = \psi\bigl(d(Tu_{n-1}, Tu_n)\bigr) \leqslant F\bigl(\psi\bigl(m(u_{n-1}, u_n)\bigr), \phi\bigl(m(u_{n-1}, u_n)\bigr)\bigr), \tag{3}$$

where

$$\begin{aligned} m(u_{n-1}, u_n) &= \max\biggl\{d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_{n+1}), \\ &\qquad \frac{d(u_{n-1}, u_n)d(u_n, u_{n+1})}{1 + d(u_{n-1}, u_n)}, \frac{d(u_{n-1}, u_n)d(u_n, u_{n+1})}{1 + d(u_n, u_{n+1})}\biggr\} \\ &\leqslant \max\bigl\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\bigr\}. \end{aligned}$$

Utilizing Remark 1 and condition (3), it follows that

$$\psi\bigl(d(u_n, u_{n+1})\bigr) < \psi\bigl(\max\bigl\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\bigr\}\bigr). \tag{4}$$

If $\max\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\} = d(u_n, u_{n+1})$, we get a contradiction. Indeed, equation (4) implies

$$\psi\bigl(d(u_n, u_{n+1})\bigr) < \psi\bigl(d(u_n, u_{n+1})\bigr), \quad \text{that is,} \quad d(u_n, u_{n+1}) < d(u_n, u_{n+1}).$$

Therefore, we get that $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$ for all $n \in \mathbb{N}$. This means that there exists $\lim_{n\to+\infty} d(x_n, x_{n+1}) = \delta \geqslant 0$. If $\delta > 0$, then from (3) and Remark 1 we get

$$\begin{aligned} \psi(\delta) &\leqslant F\biggl(\psi\biggl(\max\biggl\{\delta, \delta, \delta, \frac{\delta^2}{1+\delta^2}, \frac{\delta^2}{1+\delta^2}\biggr\}\biggr), \phi\biggl(\max\biggl\{\delta, \delta, \delta, \frac{\delta^2}{1+\delta^2}, \frac{\delta^2}{1+\delta^2}\biggr\}\biggr)\biggr) \\ &< \psi\biggl(\max\biggl\{\delta, \delta, \delta, \frac{\delta^2}{1+\delta^2}, \frac{\delta^2}{1+\delta^2}\biggr\}\biggr) = \psi(\delta), \end{aligned}$$

which is a contradiction. Hence $\lim_{n\to+\infty} d(u_n, u_{n+1}) = 0$.

Further, we show that $\lim_{n\to+\infty} d(u_n, u_{n+2}) = 0$. Firstly, we have $\alpha(u_{n-1}, u_n) \geqslant 1$, i.e., $\alpha(u_{n-1}, u_{n+1}) \geqslant 1$ because T is a triangular α-admissible mapping. Therefore, we arrive at

$$\begin{aligned} \psi\bigl(d(u_n, u_{n+2})\bigr) &= \psi\bigl(d(Tu_{n-1}, Tu_{n+1})\bigr) \\ &\leqslant F\bigl(\psi\bigl(m(u_{n-1}, u_{n+1})\bigr), \phi\bigl(m(u_{n-1}, u_{n+1})\bigr)\bigr), \end{aligned}$$

where

$$\begin{aligned} &m(u_{n-1}, u_{n+1}) \\ &\quad = \max\biggl\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n), d(u_{n+1}, u_{n+2}), \\ &\qquad \frac{d(u_{n-1}, u_n)d(u_{n+1}, u_{n+2})}{1 + d(u_{n-1}, u_{n+1})}, \frac{d(u_{n-1}, u_n)d(u_{n+1}, u_{n+2})}{1 + d(u_n, u_{n+2})}\biggr\}. \end{aligned}$$

Since there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$,

$$\frac{d(u_{n-1}, u_n) d(u_{n+1}, u_{n+2})}{1 + d(u_{n-1}, u_{n+1})} \leqslant d(u_{n-1}, u_n)$$

and

$$\frac{d(u_{n-1}, u_n) d(u_{n+1}, u_{n+2})}{1 + d(u_n, u_{n+2})} \leqslant d(u_{n-1}, u_n),$$

also, $d(u_{n-1}, u_n) \to 0$ as $n \to +\infty$, for $n \geqslant n_0$, we obtain

$$\begin{aligned} m(u_{n-1}, u_{n+1}) \leqslant \max\bigl\{& d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n), d(u_{n+1}, u_{n+2}), \\ & d(u_{n-1}, u_n) d(u_{n+1}, u_{n+2})\bigr\}, \end{aligned}$$

that is,

$$\begin{aligned} m(u_{n-1}, u_{n+1}) &\leqslant \max\bigl\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n), d^2(u_{n-1}, u_n)\bigr\} \\ &\leqslant \max\bigl\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n)\bigr\}. \end{aligned}$$

Hence, for $n \geqslant n_0$, according to Remark 1, it follows that

$$\psi\bigl(d(u_n, u_{n+2})\bigr) < \psi\bigl(\max\bigl\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n)\bigr\}\bigr).$$

This amounts to say that, for $n \geqslant n_0$,

$$d(u_n, u_{n+2}) < \max\bigl\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n)\bigr\}.$$

Now, we get

$$\begin{aligned} \limsup_{n\to+\infty} d(u_n, u_{n+2}) &\leqslant \limsup_{n\to+\infty}\bigl(\max\bigl\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n)\bigr\}\bigr) \\ &= \max\Bigl\{\limsup_{n\to+\infty} d(u_{n-1}, u_{n+1}), \limsup_{n\to+\infty} d(u_{n-1}, u_n)\Bigr\} \\ &= \max\Bigl\{\limsup_{n\to+\infty} d(u_{n-1}, u_{n+1}), 0\Bigr\} \\ &= \limsup_{n\to+\infty} d(u_{n-1}, u_{n+1}). \end{aligned}$$

Hence, we have

$$\limsup_{n\to+\infty} d(u_n, u_{n+2}) \leqslant \limsup_{n\to+\infty} d(u_{n-1}, u_{n+1}).$$

Suppose that

$$\limsup_{n\to+\infty} d(u_n, u_{n+2}) = \overline{\delta} > 0.$$

Then we acquire the following:

$$\psi\Bigl(\limsup_{n\to+\infty} d(u_n, u_{n+2})\Bigr) \leqslant F\Bigl(\psi\Bigl(\limsup_{n\to+\infty} d(u_n, u_{n+2})\Bigr), \phi\Bigl(\limsup_{n\to+\infty} d(u_n, u_{n+2})\Bigr)\Bigr),$$

that is,

$$\psi(\overline{\delta}) \leqslant F\bigl(\psi(\overline{\delta}), \phi(\overline{\delta})\bigr) < \psi(\overline{\delta}),$$

which is a contradiction. Hence, it follows that $\lim_{n\to+\infty} d(x_n, x_{n+2}) = 0$.

In order to prove that the sequence $\{u_n\}$ is a Cauchy sequence, we use Lemma 2. According to Lemma 1, $\alpha(u_{n(k)}, u_{m(k)}) \geqslant 1$ if $m(k) < n(k)$, then putting in (1) $p = u_{n(k)}$, $r = u_{m(k)}$, we obtain

$$\psi\bigl(d(u_{n(k)+1}, u_{m(k)+1})\bigr) \leqslant F\bigl(\psi\bigl(m(u_{n(k)}, u_{m(k)})\bigr), \phi\bigl(m\bigl(u_{n(k)}, u_{m(k)}\bigr)\bigr)\bigr), \tag{5}$$

where

$$\begin{aligned} m(u_{n(k)}, u_{m(k)}) = \max\biggl\{& d(u_{n(k)}, u_{m(k)}), d(u_{n(k)}, u_{n(k)+1}), d(u_{m(k)}, u_{m(k)+1}), \\ & \frac{d(u_{n(k)}, u_{n(k)+1}) d(u_{m(k)}, u_{m(k)+1})}{1 + d(u_{n(k)}, u_{m(k)})}, \\ & \frac{d(u_{n(k)}, u_{n(k)+1}) d(u_{m(k)}, u_{m(k)+1})}{1 + d(u_{n(k)+1}, u_{m(k)+1})}\biggr\} \\ \underset{k\to+\infty}{\to} \max\biggl\{& \varepsilon, 0, 0, \frac{0 \cdot 0}{1+\varepsilon}, \frac{0 \cdot 0}{1+\varepsilon}\biggr\} = \varepsilon. \end{aligned}$$

Now, since F, ψ and ϕ are continuous, taking limit $k \to +\infty$ in (5), and utilizing Remark 1, we obtain

$$\psi(\varepsilon) \leqslant F\bigl(\psi(\varepsilon), \phi(\varepsilon)\bigr) < \psi(\varepsilon),$$

which is a contradiction. Hence the sequence $\{u_n\}$ is a Cauchy sequence. Since (M, d) is a complete rectangular metric space, there exists a point $u^* \in X$ such that $u_n \to u^*$ as $n \to +\infty$. If T is continuous, we get $u_{n+1} = Tu_n \to Tu^*$ as $n \to +\infty$. Let $Tu^* \neq u^*$. Since $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$ for all $n \in \mathbb{N} \cup \{0\}$, according to Lemma 3, we conclude that u_n's are distinct. Therefore, there exists $n_1 \in \mathbb{N}$ such that $u^*, Tu^* \notin \{u_n\}_{n \geqslant n_1}$. Further, by (iii) it follows that

$$d(u^*, Tu^*) \leqslant d(u^*, u_n) + d(u_n, u_{n+1}) + d(u_{n+1}, Tu^*)$$

whenever $n \geqslant n_1$. Taking the limit in the last inequality, it follows that $d(u^*, Tu^*) = 0$, i.e., $u^* = Tu^*$, which is again a contradiction.

In the case that (M, d) is α-regular, and since $\alpha(u_n, u^*) \geqslant 1$ for all $n \in \mathbb{N}$, from (1) we obtain the following:

$$\psi\bigl(d(Tu_n, Tu^*)\bigr) \leqslant F\bigl(\psi\bigl(m(u_n, u^*)\bigr), \phi\bigl(m(u_n, u^*)\bigr)\bigr), \tag{6}$$

where

$$\begin{aligned} m(u_n, u^*) = \max\biggl\{& d(u_n, u^*), d(u_n, u_{n+1}), d(u^*, Tu^*), \\ & \frac{d(u_n, u_{n+1}) d(u^*, Tu^*)}{1 + d(u_n, u^*)}, \frac{d(u_n, u_{n+1}) d(u^*, Tu^*)}{1 + d(u_{n+1}, Tu^*)}\biggr\} \\ \underset{n\to+\infty}{\to}\ & d(u^*, Tu^*). \end{aligned}$$

Passing limit $n \to +\infty$ in (6) and using Proposition 1, the continuity of the functions F, ψ, and ϕ as well as Remark 1, it follows that if $u^* \neq Tu^*$, then

$$\psi\big(d(u^*, Tu^*)\big) \leqslant F\big(\psi\big(d(u^*, Tu^*)\big), \phi\big(d(u^*, Tu^*)\big)\big) < \psi\big(d(u^*, Tu^*)\big),$$

which is a contradiction. Hence, u^* is a fixed point of T.

Now, we show that the fixed point is unique if $\alpha(p, r) \geqslant 1$ whenever $p, r \in \operatorname{Fix}(T)$. Indeed, in this case, by the contractive condition (1), for such possible distinct fixed points p, r, we have

$$\psi\big(d(p, r)\big) = \psi\big(d(Tp, Tr)\big) \leqslant F\big(\psi\big(m(p, r)\big), \phi\big(m(p, r)\big)\big), \tag{7}$$

where

$$\begin{aligned} m(p, r) &= \max\left\{d(p, r), d(p, Tp), d(r, Tr), \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)}, \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}\right\} \\ &= \max\left\{d(p, r), 0, 0, \frac{0 \cdot 0}{1 + 0}, \frac{0 \cdot 0}{1 + 0}\right\} = d(p, r). \end{aligned}$$

Hence, (7) becomes

$$\psi\big(d(p, r)\big) \leqslant F\big(\psi\big(d(p, r)\big), \phi\big(d(p, r)\big)\big) < \psi\big(d(p, r)\big),$$

which is again a contradiction. The proof of the Theorem 3 is complete. □

Remark 2. In the proof of the main theorem [7, p. 586, line 1_-, Thm. 1, case 3], the authors used the fact that rectangular metric d (see also inequality (5) on the same page) is continuous, which is not given in the formulation of Theorem 1 in [7]. In the proof of the same theorem (p. 585), the authors also claimed that $\lim_{n\to+\infty} d(u_n, u_{n+2}) = 0$, which is not correct because we do not know whether $\lim_{n\to+\infty} d(u_n, u_{n+2})$ exists or not.

By taking $F(x, y) = x - y$ in Theorem 3, we obtain the following result as a corollary.

Corollary 4. *Let (M, d) be a complete rectangular metric space, and let $T : M \to M$ be a triangular α-admissible mapping. Assume that there exist $\psi, \phi \in \Psi$ such that, for $p, r \in M$,*

$$\alpha(p, r) \geqslant 1 \quad \textit{implies} \quad \psi\big(d(Tp, Tr)\big) \leqslant \psi\big(m(p, r)\big) - \phi\big(m(p, r)\big),$$

where

$$m(p, r) = \max\left\{d(p, r), d(p, Tp), d(r, Tr), \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)}, \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}\right\}.$$

Also suppose that the following assertions hold:

(i) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geqslant 1$;*
(ii) *T is continuous or (M, d) is α-regular.*

Then T has a fixed point. Moreover, if for all $p, r \in F(T)$, we have $\alpha(p, r) \geqslant 1$, then the fixed point is unique.

Taking $\psi(t) = 1$ for all $t \in [0, +\infty)$ in Corollary 4, the following useful corollary is obtained.

Corollary 5. *Let (M, d) be a complete rectangular metric space, and let $T : M \to M$ be a triangular α-admissible mapping. Assume that there exists $\phi \in \Psi$ such that, for $p, r \in M$,*

$$\alpha(p, r) \geqslant 1 \quad \textit{implies} \quad d(Tp, Tr) \leqslant m(p, r) - \phi\bigl(m(p, r)\bigr),$$

where

$$m(p, r) = \max\left\{d(p, r), d(p, Tp), d(r, Tr), \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)}, \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}\right\}.$$

Suppose also that the following assertions hold:

(i) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geqslant 1$;*
(ii) *T is continuous or (M, d) is α-regular.*

Then T has a fixed point. Moreover, if for all $p, r \in F(T)$, we have $\alpha(p, r) \geqslant 1$, then the fixed point is unique.

Consider $\phi(t) = (1 - q)t$ for $0 < q < 1$ in Corollary 5, then we obtain the following result.

Corollary 6. *Let (M, d) be a complete rectangular metric space, and let $T : M \to M$ be a triangular α-admissible mapping such that, for $p, r \in M$,*

$$\alpha(p, r) \geqslant 1 \quad \textit{implies} \quad d(Tp, Tr) \leqslant q \cdot m(p, r),$$

where

$$m(p, r) = \max\left\{d(p, r), d(p, Tp), d(r, Tr), \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)}, \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}\right\}.$$

Also suppose that the following conditions are contended:

(i) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geqslant 1$;*
(ii) *T is continuous or (M, d) is α-regular.*

Then T has a fixed point. Moreover, if for all $p, r \in F(T)$, we have $\alpha(p, r) \geqslant 1$, then the fixed point is unique.

3 Application to a dynamical programming

This section aims to apply our results to solve the existence and uniqueness of the solution of the dynamic programming problem. In particular, the problem of dynamic programming related to multistage process reduces to solving the existence and uniqueness of the solution of the following functional equation:

$$q(x) = \sup_{y \in D}\bigl\{f(x, y) + G\bigl(x, y, q\bigl(\rho(x, y)\bigr)\bigr)\bigr\} \quad \text{for all } x \in W, \tag{8}$$

where $\rho : W \times D \to W$, $f : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$. We suppose that $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, U and V are Banach spaces. Let $B(W)$ denote the set of all bounded real valued functions on W, and for an arbitrary $h \in B(W)$, define $||h||_\infty = \sup_{x\in W} |h(x)|$. Clearly, the pair $(B(W), \|\cdot\|_\infty)$ is a Banach space. For details, see [18, 19].

In fact, the distance in $B(W)$ is given by

$$d_\infty(h,k) = \sup_{x\in W} \bigl|h(x) - k(x)\bigr| \quad \text{for all } h,k \in B(W).$$

Define $T : B(W) \to B(W)$ by

$$T(h)(x) = \sup_{y\in D} \bigl\{ f(x,y) + G\bigl(x,y,h\bigl(\rho(x,y)\bigr)\bigr) \bigr\} \quad \text{for all } h \in B(W),\ x \in W. \tag{9}$$

Obviously, T is well-defined if the functions f and G are bounded.

Theorem 4. *Let T be an operator defined by* (9)*, and suppose that the following conditions hold*:

(i) *$G : W \times D \times \mathbb{R} \to \mathbb{R}$ and $f : W \times D \to \mathbb{R}$ are continuous and bounded*;
(ii) *there exist $F \in \mathcal{C}$ and $\psi, \phi \in \Psi$ such that $\psi(t) = t$ for all $t \in [0,\infty)$, and*

$$\bigl|G\bigl(x,y,h\bigl(\rho(x,y)\bigr)\bigr) - G\bigl(x,y,k\bigl(\rho(x,y)\bigr)\bigr)\bigr| \leqslant F\bigl(\psi\bigl(m(h,k)\bigr), \phi\bigl(m(h,k)\bigr)\bigr),$$

where

$$m(h,k) = \max\biggl\{ d_\infty(h,k), d_\infty(h,Th), d_\infty(k,Tk), \frac{d_\infty(h,Th)d_\infty(k,Tk)}{1+d_\infty(h,k)}, \frac{d_\infty(h,Th)d_\infty(k,Tk)}{1+d_\infty(Th,Tk)} \biggr\}$$

for all $h,k \in B(W)$, $x \in W$ and $y \in D$.

Then the functional equation (8) *has a unique solution.*

Proof. Let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$\begin{aligned}
T(h_1)(x) &< f(x,y_1) + G\bigl(x,y_1,h_1\bigl(\rho(x,y_1)\bigr)\bigr) + \lambda,\\
T(h_2)(x) &< f(x,y_2) + G\bigl(x,y_2,h_2\bigl(\rho(x,y_2)\bigr)\bigr) + \lambda,\\
T(h_1)(x) &\geqslant f(x,y_2) + G\bigl(x,y_2,h_1\bigl(\rho(x,y_2)\bigr)\bigr),\\
T(h_2)(x) &\geqslant f(x,y_1) + G\bigl(x,y_1,h_2\bigl(\rho(x,y_1)\bigr)\bigr),
\end{aligned}$$

which yields

$$\begin{aligned}
&T(h_1)(x) - T(h_2)(x)\\
&\quad < G\bigl(x,y_1,h_1\bigl(\rho(x,y_1)\bigr)\bigr) - G\bigl(x,y_1,h_2\bigl(\rho(x,y_1)\bigr)\bigr) + \lambda\\
&\quad \leqslant \bigl|G\bigl(x,y_1,h_1\bigl(\rho(x,y_1)\bigr)\bigr) - G\bigl(x,y_1,h_2\bigl(\rho(x,y_1)\bigr)\bigr)\bigr| + \lambda.
\end{aligned}$$

In the same manner, we acquire

$$T(h_2)(x) - T(h_1)(x) \leqslant \left|G\big(x, y_1, h_2\big(\rho(x, y_1)\big)\big) - G\big(x, y_1, h_1\big(\rho(x, y_1)\big)\big)\right| + \lambda.$$

Since λ is arbitrary, we conclude

$$\left|T(h_1)(x) - T(h_2)(x)\right| \leqslant \left|G\big(x, y_1, h_1\big(\rho(x, y_1)\big)\big) - G\big(x, y_1, h_2\big(\rho(x, y_1)\big)\big)\right|.$$

This amounts to say that

$$\psi\big(d_\infty\big(T(h_1), T(h_2)\big)\big) = d_\infty\big(T(h_1), T(h_2)\big) \leqslant F\big(\psi\big(m(h_1, h_2)\big), \phi\big(m(h_1, h_2)\big)\big),$$

where

$$\begin{aligned} m(h_1, h_2) = \max\bigg\{ & d_\infty(h_1, h_2), d_\infty(h_1, Th_1), d_\infty(h_2, Th_2), \\ & \frac{d_\infty(h_1, Th_1) d_\infty(h_2, Th_2)}{1 + d_\infty(h_1, h_2)}, \frac{d_\infty(h_1, Th_1) d_\infty(h_2, Th_2)}{1 + d_\infty(Th_1, Th_2)} \bigg\}. \end{aligned}$$

Lastly, we specify $\alpha : B(W) \times B(W) \to [0, \infty)$ such that

$$\alpha(h_1, h_2) = \begin{cases} 1, & h_1, h_2 \in B(W), \\ 0 & \text{otherwise}. \end{cases}$$

Evidently, $\alpha(h_1, h_2) = 1$ and $\alpha(T(h_1), T(h_2)) = 1$ for all $h_1, h_2 \in B(W)$. This endorses that T is a triangular α-admissible mapping. Hence, due to Theorem 3, T has a unique fixed point $h^* \in B(W)$, that is, h^* is a unique solution of the functional equation (8). This completes the proof. □

4 Conclusion

This article is devoted to addressing some weaknesses of the main results introduced in [7]. Antithetical to the results in [7], we used only one function $\alpha : M \times M \to [0, +\infty)$ instead of two α and η as used in [8, Defs. 2.3, 3.1]. We also dropped the property of Hausdorffness of the rectangular metric space (M, d) and the continuity of the mapping d. Using our new approach, we proved that a Picard sequence is Cauchy in the framework of rectangular metric spaces. Our obtained results complement and enrich several methods in the existing state-of-art. Thereafter, we apply our results to study a dynamic programming problem associated with a multistage process to affirm the applicability of the obtained results.

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References

1. H.H. Alsulami, S. Chandok, M.-A. Taoudi, İ.M. Erhan, Some fixed point theorems for (α-ψ)-rational type contractive mappings, *Fixed Point Theory Appl.*, **2015**:97, 2015, https://doi.org/10.1186/s13663-015-0332-3.
2. A.H. Ansari, Note on $\phi-\psi$-contractive type mappings and related fixed point, in *Proceedings of the Second Regional Conference on Mathematical Sciences and Applications, Tonekabon, Iran, September, 2014*, Payame Noor University, 2014, pp. 377–380.
3. H. Aydi, C.-M. Chen, E. Karapınar, Interpolative ćirić-reich-rus type contractions via the Branciari distance, *Mathematics*, **7**:84, 2019, https://doi.org/10.3390/math7010084.
4. H. Aydi, E. Karapınar, B. Samet, Fixed points for generalized (α, ψ)-contractions on generalized metric spaces, *J. Inequal. Appl.*, **2014**:229, 2014, https://doi.org/10.1186/1029-242X-2014-229.
5. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Math.*, **3**:133–181, 1922, https://doi.org/10.4064/fm-3-1-133-181.
6. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math.*, **57**(1–2):31–37, 2000.
7. L. Budhia, H. Aydi, A.H. Ansari, D. Gopal, Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations, *Nonlinear Anal. Model. Control*, **25**(4):580–597, 2020, https://doi.org/10.15388/namc.2020.25.17928.
8. L. Budhia, M. Kir, D. Gopal, H. Kiziltunç, New fixed point results in rectangular metric space and application to fractional calculus, *Tbilisi Math. J.*, **10**(1):91–104, 2017, https://doi.org/10.1515/tmj-2017-0006.
9. P. Collaço, J.C.E. Silva, A complete comparison of 25 contraction conditions, *Nonlinear Anal., Theory Methods Appl.*, **30**(1):471–476, 1997, https://doi.org/10.1016/S0362-546X(97)00353-2.
10. P. Das, L.K. Dey, Fixed point of contractive mappings in generalized metric spaces, *Math. Slovaca*, **59**(4):499–504, 2009, https://doi.org/10.2478/s12175-009-0143-2.
11. İ.M. Erhan, E. Karapınar, T. Sekulić, Fixed points of (ψ, ϕ) contractions on rectangular metric spaces, *Fixed Point Theory Appl.*, **2012**:138, 2012, https://doi.org/10.1186/1687-1812-2012-138.
12. Z. Kadelburg, S. Radenović, Fixed point results in generalized metric spaces without Hausdorff property, *Math. Sci.*, **8**(2):125, 2014, https://doi.org/10.1007/s40096-014-0125-6.
13. E. Karapınar, Discussion on $\alpha-\psi$ contractions on generalized metric spaces, *Abstr. Appl. Anal.*, **2014**:962784, 2014, https://doi.org/10.1155/2014/962784.
14. E. Karapınar, P. Kumam, P. Salimi, On α-ψ-Meir-Keeler contractive mappings, *Fixed Point Theory Appl.*, **2013**:94, 2013, https://doi.org/10.1186/1687-1812-2013-94.
15. M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30**(1):1–9, 1984, https://doi.org/10.1017/S0004972700001659.

16. L. Kikina, K. Kikina, On fixed point of a Ljubomir Ciric quasi-contraction mapping in generalized metric spaces, *Publ. Math.*, **83**(3):353–358, 2013.

17. W.A. Kirk, N. Shahzad, Generalized metrics and Caristi's theorem, *Fixed Point Theory and Applications*, **2013**(1):1–9, 2013, https://doi.org/10.1186/1687-1812-2013-129.

18. Z. Liu, Existence theorems of solutions for certain classes of functional equations arising in dynamic programming, *J. Math. Anal. Appl.*, **262**(2):529–553, 2001, https://doi.org/10.1006/jmaa.2001.7551.

19. Z. Liu, Y. Xu, J.S. Ume, S.M. Kang, Solutions to two functional equations arising in dynamic programming, *J. Comput. Appl. Math.*, **192**(2):251–269, 2006, https://doi.org/10.1016/j.cam.2005.04.033.

20. N. Lu, F. He, S. Li, A note on the paper "A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable", *J. Fixed Point Theory Appl.*, **22**(4):96, 2020, https://doi.org/10.1007/s11784-020-00831-3.

21. B. Mohammadi, Sh. Rezapour, On modified α-ψ-contractions, *J. Adv. Math. Stud.*, **6**(2):162–163, 2013.

22. S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, Some results on weakly contractive maps, *Bull. Iran. Math. Soc.*, **38**(3):625–645, 2012.

23. P. Salimi, A. Latif, N. Hussain, Modified α-ψ-contractive mappings with applications, *Fixed Point Theory Appl.*, **2013**:151, 2013, https://doi.org/10.1186/1687-1812-2013-151.

24. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha{-}\psi$-contractive type mappings, *Nonlinear Anal., Theory Methods Appl.*, **75**(4):2154–2165, 2012, https://doi.org/10.1016/j.na.2011.10.014.

25. M. Younis, D. Singh, I. Altun, V. Chauhan, Graphical structure of extended b-metric spaces: An application to the transverse oscillations of a homogeneous bar, *Int. J. Nonlinear Sci. Numer. Simul.*, 2021, https://doi.org/10.1515/ijnsns-2020-0126.

26. M. Younis, D. Singh, A. Goyal, A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable, *J. Fixed Point Theory Appl.*, **21**(1):33, 2019, https://doi.org/10.1007/s11784-019-0673-3.

27. M. Younis, D. Singh, A. Petruşel, Applications of graph kannan mappings to the damped spring-mass system and deformation of an elastic beam, *Discrete Dynamics in Nature and Society*, **2019**, 2019, https://doi.org/10.1155/2019/1315387.

28. M. Younis, D. Singh, S. Radenović, M. Imdad, Convergence theorems for generalized contractions and applications, *Filomat*, **34**(3):945–964, 2020, https://doi.org/10.2298/FIL2003945Y.