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The Error Estimates of Kronrod Extension for Gauss-Radau and Gauss-Lobatto Quadrature with the Four Chebyshev Weights

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Abstract. In this paper, we consider the Kronrod extension for the Gauss-Radau and Gauss-Lobatto quadrature consisting of any one of the four Chebyshev weights. The main purpose is to effectively estimate the error of these quadrature formulas. This estimate needs a calculation of the maximum of the modulus of the kernel. We compute explicitly the kernel function and determine the locations on the ellipses where a maximum modulus of the kernel is attained. Based on this, we derive effective error bounds of the Kronrod extensions if the integrand is an analytic function inside of a region bounded by a confocal ellipse that contains the interval of integration.

1. Introduction

1.1. *Gauss Kronrod quadrature formula*

Let us consider the Gauss-Kronrod quadrature formula for the (nonnegative) weight function w on the interval $[-1,1]$:

$$\int_{-1}^{1} f(t)w(t)\,dt = Q_n(f) + R_n(f), \quad Q_n(f) = \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^* f(\tau_\mu^*), \tag{1}$$

where τ_ν are the zeros of the n-th degree (monic) orthogonal polynomial $\pi_n(\cdot)$ relative to the weight function w, i.e., τ_ν are the nodes of the corresponding n-point Gaussian type quadrature formula G_n relative to the weight function w,

$$G_n(f) = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu),$$

which has the algebraic degree of precision $2n-1$. The nodes τ_μ^* and all weights σ_ν, σ_μ^* are chosen such that formula (1) has maximum degree of precision at least $3n+1$, i.e., $R_n(f) = 0$ for all $f \in \mathcal{P}_{3n+1}$. It is known that the τ_μ^* are zeros of a (monic) polynomial $\pi_{n+1}^*(t) = \pi_{n+1}(t, w^{KG})$ (where $w^{KG} = \pi_n(t; w^G)w^G(t)$ and $w^G(t) = w$), of degree $n+1$, called Stieltjes polynomial, which satisfies the orthogonality condition

$$\int_{-1}^{1} \pi_{n+1}^*(t)\pi_n(t)p(t)\,w(t)\,dt = 0, \quad p \in \mathcal{P}_n.$$

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1.2. Kronrod extension to generalized Gauss-Radau and Gauss-Lobatto formula

Shikang Li applied Kronrod's idea to generalized Gauss-Radau and Gauss-Lobatto formula with double end point (see [7]), with an assumption of that weight function σ is one of the four Chebyshev weights:

(a) $w_1(t) = (1-t^2)^{-1/2}$, (b) $w_2(t) = (1-t^2)^{1/2}$,

(c) $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2}$, (d) $w_4(t) = (1-t)^{1/2}(1+t)^{-1/2}$;

The Kronrod extension for the generalized Gauss-Radau quadrature rule has the form

$$\int_{-1}^{1} f(t)w(t)dt = \sigma_0 f(-1) + \sigma_0' f'(-1) + \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^* f(\tau_\mu^*) + R_n^{KR}(f) \tag{2}$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of $\pi_n(\cdot;(1+t)^2 w(t))$, the n-th degree monic orthogonal polynomial relative to the weight $w^R(t) = (1+t)^2 w(t)$, and $\tau_\mu^* = \tau_{\mu,n}^*$, $\sigma_0, \sigma_0', \sigma_\nu$ and σ_μ^* are chosen so that formula (2) has maximum degree of precision $\geqslant 3n+3$. It's established that τ_μ^* must be zeros of the monic orthogonal polynomial π_{n+1}^* of degree $n+1$ known as Stieltjes polynomial, orthogonal to all polynomials of lower degree in the sense

$$\int_{-1}^{1} \pi_{n+1}^*(t)p(t)\pi_n(t;w^R(t))w^R(t)dt = 0, \quad p \in \mathcal{P}_n,$$

with respect to the weight function $w^{KR}(t) = \pi_n(t;w^R(t))w^R(t)$.
Analogously exists an optimal extension of the Gauss-Lobatto formula with double end points associated with the weight function w:

$$\int_{-1}^{1} f(t)w(t)dt = \sigma_0 f(-1) + \sigma_0' f'(-1) + \sigma_{n+1} f(1) + \sigma_{n+1}' f'(1) + \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^* f(\tau_\mu^*) + R_n^{KL}(f) \tag{3}$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of $\pi_n(\cdot;(1-t^2)^2 w(t))$, and $\tau_\mu^* = \tau_{\mu,n}^*$ are the zeros of π_{n+1}^* witch is orthogonal to all polynomials of degree $\leqslant n$ in the sense

$$\int_{-1}^{1} \pi_{n+1}^*(t)p(t)\pi_n(t;w^L(t))w^L(t)dt = 0, \quad p \in \mathcal{P}_n,$$

with respect to the weight function $w^{KL}(t) = \pi_n(t;w^L(t))w^L(t)$.
π_{n+1} uniquely exists in both cases (2) and (3). The question of fact that all nodes are real and inside interval $[-1,1]$, as well as a question of an interlacing of the nodes τ_ν and τ_μ^*, and also positivity of the weights have been studied in [7].
First, we give some relations among Chebyshev polynomials of n-th degree, which we use later in the error estimation.

$$T_n(\cos(\theta)) = \cos(n\theta), \quad U_n(\cos(\theta)) = \frac{\sin(n+1)\theta}{\sin(\theta)} \tag{4}$$

$$V_n(\cos(\theta)) = \frac{\cos((n+\frac{1}{2})\theta)}{\cos(\theta/2)}, \quad W_n(\cos(\theta)) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\theta/2)} \tag{5}$$

Where $T_0 = U_0 = V_0 = W_0 = 1$ and $T_1(t) = t, U_1(t) = 2t, V_1(t) = 2t-1, W_1(t) = 2t+1$ and $t = \cos(\theta)$. They all satisfy the same recurrence relation $y_{n+1} = 2ty_n - y_{n-1}, n = 1,2,3....$ And relation among these polynomials, which we use later constantly in the further development, are given.

$$T_n T_m = \cos(n\theta)\cos(m\theta) = \frac{1}{2}(\cos((n+m)\theta) + \cos((n-m)\theta)) = \frac{1}{2}(T_{n+m} + T_{|n-m|}), \tag{6}$$

$$U_nU_m = \frac{\sin((n+1)\theta)\sin((m+1)\theta)}{\sin(\theta)\cdot\sin(\theta)} = \frac{\frac{1}{2}(\cos(n-m)\theta - \cos(n+m+2)\theta)}{\sin^2\theta} = \frac{1}{2}\frac{(T_{|n-m|} - T_{n+m+2})}{1-t^2}, \tag{7}$$

$$V_nV_m = \frac{\cos((n+\frac{1}{2})\theta)\cos((m+\frac{1}{2})\theta)}{\cos(\frac{1}{2}\theta)\cdot\cos(\frac{1}{2}\theta)} = \frac{\frac{1}{2}(\cos(n-m)+\cos(n+m+1))}{\cos^2\frac{\theta}{2}} = \frac{(T_{|n-m|} + T_{n+m+1})}{1+t}, \tag{8}$$

$$U_nT_m = \frac{\sin((n+1)\theta)\cos(m\theta)}{\sin(\theta)} = \begin{cases} \frac{1}{2}\left(U_{n+m} + U_{n-m}\right), & n+1 > m \\ \frac{1}{2}U_{2n+1}, & n+1 = m \\ \frac{1}{2}\left(U_{n+m} - U_{m-n-2}\right), & n+1 < m. \end{cases} \tag{9}$$

1.3. The orthogonal polynomials $\pi_n(t, w)$ with respect to the Chebyshev weight functions

For the Gauss-Radau formula with double end point at -1 weight function is $w_i^R(t) = (1+t)^2 w_i(t)$ and w_i is one of Chebyshev weights. Also, for Gauss-Lobatto formula with double end point weight function is $w_i^L(t) = (1-t^2)^2 w_i(t)$. The nodal polynomial is $\pi_n^{R,i}(t) = \pi_n(t; w_i^R))$ and $\pi_n^{L,i}(t) = \pi_n(t; w_i^L))$ associated with the internal nodes in the generalized Gauss-Radau and Gauss-Lobatto formula. In the context where the meaning is clear, we will denote these polynomials with $\pi_n(t)$.
Gautschi and Li developed for all four Chebyshev weight functions both Gauss-Radau and Gauss-Lobatto formulae having end points of multiplicity 2 in [2]. The explicit formulas, which we used for further research, can easily be derived from [2] and are presented below.

1.3.1. Gauss-Radau nodal polynomial for the weight function $w_1^R = (1-t)^{-1/2}(1+t)^{3/2}$

$$\pi_n^{R,1}(t) = \frac{1}{2^{n+1}(1+t)^2}\left\{T_{n+2}(t) + 4\frac{n+1}{2n+1}T_{n+1}(t) + \frac{2n+3}{2n+1}T_n(t)\right\} \tag{10}$$

or

$$\pi_n^{R,1}(t) = \frac{1}{2^{n+1}(1+t)}\left\{V_{n+1}(t) + \frac{2n+3}{2n+1}V_n(t)\right\}.$$

1.3.2. Gauss-Radau nodal polynomial for the weight function $w_2^R = (1-t)^{1/2}(1+t)^{5/2}$

$$\pi_n^{R,2}(t) = \frac{1}{2^{n+2}(1+t)^2}\left\{U_{n+2}(t) + 4\frac{n+3}{2n+3}U_{n+1}(t) + \frac{(n+3)(2n+5)}{(n+1)(2n+3)}U_n(t)\right\}. \tag{11}$$

1.3.3. Gauss-Radau nodal polynomial for the weight function $w_3^R = (1-t)^{-1/2}(1+t)^{5/2}$

$$\pi_n^{R,3}(t) = \frac{1}{2^{n+2}(1+t)^2}\left\{V_{n+2}(t) + \frac{2n+5}{n+1}V_{n+1}(t) + \frac{(n+2)(2n+5)}{(n+1)(2n+1)}V_n(t)\right\}. \tag{12}$$

1.3.4. Gauss-Radau nodal polynomial for the weight function $w_4^R = (1-t)^{1/2}(1+t)^{3/2}$

$$\pi_n^{R,4}(t) = \frac{1}{2^{n+1}(1+t)}\left\{U_{n+1}(t) + \frac{n+2}{n+1}U_n(t)\right\}. \tag{13}$$

1.3.5. Gauss-Lobatto nodal polynomial for the weight function $w_1^L = (1-t^2)^{3/2}$

$$\pi_n^{L,1}(t) = \frac{1}{2^{n+2}} \frac{1}{t^2-1} \left\{ U_{n+2}(t) - \frac{n+3}{n+1} U_n(t) \right\}. \tag{14}$$

1.3.6. Gauss-Lobatto nodal polynomial for the weight function $w_2^L = (1-t^2)^{5/2}$

$$\pi_n^{L,2}(t) = \frac{1}{2^{n+4}} \frac{1}{(1-t^2)^2} \left\{ U_{n+4}(t) - 2\frac{n+5}{n+2} U_{n+2}(t) + \frac{(n+4)(n+5)}{(n+1)(n+2)} U_n(t) \right\}. \tag{15}$$

1.4. Stieltjes polynomials $\pi_{n+1}^*(t,w)$ for Gauss-Radau and Gauss-Lobatto formula

Given an orthogonal polynomial $\pi_n(\cdot) = \pi_n(\cdot; w)$ of degree n with respect to a weight function w on $[-1,1]$, there is associated with it a unique (monic) polynomial $\pi_{n+1}^*(\cdot) = \pi_{n+1}^*(\cdot; w)$ of degree $n+1$, called Stieltjes polynomial, explained before, and for Kronrod-Radau $\pi_{n+1}^*(t) = \pi_{n+1}(t; w^{KR})$ where $w^{KR}(t) = \pi_n(t, w^R) w^R(t)$ with $w^R = (1+t^2)w(t)$ while Kronrod-Lobatto $\pi_{n+1}^*(t) = \pi_{n+1}(t; w^{KL})$ where $w^{KL}(t) = \pi_n(t, w^L) w^L(t)$ with $w^L = (1-t^2)^2 w(t)$. In this section, we express these Stieltjes polynomials. The explicit formulas for $\pi_{n+1}^*(t,w)$ were calculated in [7] and are available for all the cases we considered here.

1.4.1. Stieltjes polynomials $\pi_{n+1}^*(t,w)$ for Gauss-Radau formula and weight function w_1^R

$$\pi_{n+1}^*(t) = \frac{1}{2^n} \left\{ T_{n+1}(t) + 4\frac{n+1}{2n+3} \sum_{k=0}^{n-1} (-1)^{k+1} \left(\frac{2n+1}{2n+3} \right)^k T_{n-k}(t) + (-1)^{n+1} \frac{2(n+1)}{2n+3} \left(\frac{2n+1}{2n+3} \right)^n T_0 \right\} \tag{16}$$

where T_k is the first kind Chebyshev polynomial of degree k. Or

$$\pi_{n+1}^*(t) = \frac{1}{2^n} \left\{ V_{n+1}(t) - \frac{2n+1}{2n+3} V_n(t) - \frac{8(n+1)}{(2n+3)^2} \sum_{k=1}^{n} \left(-\frac{2n+1}{2n+3} \right)^{k-1} V_{n-k}(t) \right\}.$$

1.4.2. Stieltjes polynomials $\pi_{n+1}^*(t,w)$ for Gauss-Radau formula and weight function w_2^R

$$\pi_{n+1}^*(t) = \frac{1}{2^{n+1}} \left\{ U_{n+1}(t) + \sum_{k=0}^{n} c_{n-k} U_{n-k}(t) \right\} \tag{17}$$

where

$$c_{n-k} = a_0 \cos k\theta + a_1 \sin k\theta, \tag{18}$$

the coefficients a_0 and a_1 are

$$a_0 = -\frac{4(n+1)}{2n+5}, \tag{19}$$

$$a_1 = \frac{2}{\sqrt{3}(2n+5)} \cdot \left(\frac{\sqrt{(2n+3)(2n+5)}(4n^3+14n^2-2n-21)}{(n+3)(2n+5)} - 4(n+1)\sqrt{(n+1)(n+3)} \right) \tag{20}$$

and angle θ satisfying the relationship

$$\cos\theta = -2\sqrt{\frac{(n+1)(n+3)}{(2n+3)(2n+5)}}, \quad 0 < \theta < \pi.$$

1.4.3. Stieltjes polynomials $\pi^*_{n+1}(t,w)$ for Gauss-Radau formula and weight function w_3^R

$$\pi^*_{n+1}(t) = \frac{1}{2^{n+1}}\left\{V_{n+1}(t) + \sum_{k=0}^{n} c_{n-k}V_{n-k}(t)\right\} \tag{21}$$

where

$$c_{n-k} = a_0 \cos k\theta + a_1 \sin k\theta, \tag{22}$$

the coefficients a_0 and a_1 are

$$a_0 = -\frac{2n+1}{n+2}, \tag{23}$$

$$a_1 = \frac{1}{\sqrt{3}(n+2)} \cdot \left(\frac{2(n+1)^{3/2}(4n^2+4n-17)}{(n+2)^{1/2}(2n+5)} - (2n+1)\sqrt{(2n+5)(2n+1)}\right) \tag{24}$$

and angle θ satisfying the relationship

$$cos\theta = -\frac{1}{2}\sqrt{\frac{(2n+5)(2n+1)}{(n+1)(n+2)}}, \quad 0 < \theta < \pi. \tag{25}$$

1.4.4. Stieltjes polynomials $\pi^*_{n+1}(t,w)$ for Gauss-Radau formula and weight function w_4^R

$$\pi^*_{n+1}(t) = \frac{1}{2^n}\left\{T_{n+1}(t) + \sum_{k=1}^{n}\left(-\frac{n+1}{n+2}\right)^{n-k+1} T_k(t) + \frac{1}{2}\left(-\frac{n+1}{n+2}\right)^{n+1} T_0\right\} \tag{26}$$

where T_k is the first kind Chebyshev polynomial of degree k. Or

$$\pi^*_{n+1}(t) = \frac{1}{2^{n+1}}\left\{U_{n+1}(t) - \frac{n+1}{n+2}U_n(t) - \frac{2n+3}{(n+2)^2}\sum_{k=1}^{n}\left(-\frac{n+1}{n+2}\right)^{k-1} U_{n-k}(t)\right\}.$$

1.4.5. Stieltjes polynomials $\pi^*_{n+1}(t,w)$ for Gauss-Lobatto formula and weight function w_1^L

$$\pi^*_{n+1}(t) = \frac{1}{2^n}\left\{T_{n+1}(t) + \sum_{k=1}^{(n-1)/2}\left(\frac{n+1}{n+3}\right)^{k} T_{n-2k+1}(t) + \frac{1}{2}\left(\frac{n+1}{n+3}\right)^{(n+1)/2} T_0(t)\right\} \tag{27}$$

when n is odd, and

$$\pi^*_{n+1}(t) = \frac{1}{2^n}\left\{T_{n+1}(t) + \sum_{k=1}^{n/2}\left(\frac{n+1}{n+3}\right)^{k} T_{n-2k+1}(t)\right\} \tag{28}$$

when n is even.

1.4.6. Stieltjes polynomials $\pi^*_{n+1}(t,w)$ for Gauss-Lobatto formula and weight function w_2^L

$$\pi^*_{n+1}(t) = \frac{1}{2^{n+1}}\left\{U_{n+1}(t) + \sum_{k=1}^{\lfloor n/2\rfloor} c_{n-2k+1}U_{n-2k+1}(t)\right\} + \frac{1+(-1)^{n-1}}{4}c_0U_0(t), \tag{29}$$

where

$$c_{n-2k+1} = \left[a_0\left(1+\sqrt{\frac{2n+6}{(n+5)(n+2)}}\right)^k + a_1\left(1-\sqrt{\frac{2n+6}{(n+5)(n+2)}}\right)^k\right]\left(\frac{n+2}{n+4}\right)^k,$$

$$a_0 = \frac{n\sqrt{(2n+6)(n+5)} - 4\sqrt{n+2}}{2\sqrt{n+2}\left(2n+6+\sqrt{(2n+6)(n+5)(n+2)}\right)},$$

$$a_1 = \frac{4\sqrt{n+2} + n\sqrt{(2n+6)(n+5)}}{2\sqrt{n+2}\left(\sqrt{(2n+6)(n+5)(n+2)} - 2n - 6\right)}.$$

2. The error bounds of Gauss-Kronrod quadrature formulae

Here, we consider error estimation of quadrature formula (1) (special cases (2) and (3)) with multiple end nodes for analytic function. The error bounds, for integrands f having an analytic extension in a domain containing $[-1,1]$, can be obtained by a contour integration method, well established and described in ([3], [4], [5], [6], [8], [9], [10], [12], [13], [14]). These papers considered integrands f having an analytic extension into a domain Γ, which encompasses the interval $[-1,1]$, where Γ is a simple closed curve in the complex plain. The most common contours are concentric circles or confocal ellipses. Here, we consider confocal ellipses ε_p with foci at the points ∓ 1 and sum of semi-axes $\rho > 1$:

$$\varepsilon_p = \left\{z \in \mathbb{C} \quad | \quad z = \frac{1}{2}\left(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}\right), \quad 0 \leqslant \theta \leqslant 2\pi\right\}. \tag{30}$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1,1]$, while with increasing ρ it becomes more and more circle-like. There is an advantage of elliptical contours, compared to circular ones, it needs the analyticity of f on a smaller region of the complex plane. A contour integral representation for the remainder term $R_n(f)$ in a quadrature formula with multiple nodes is:

$$R_n(f) = \frac{1}{2\pi i}\oint_\Gamma K_n(z;w)f(z)\,dz, \tag{31}$$

where the *kernel* is given by

$$K_n(z) \equiv K_n(z;w) \equiv K_n^{GK}(z;w) = \frac{\varrho_n(z)}{\pi_n(z)\pi^*_{n+1}(z)}, \quad z \notin [-1,1], \tag{32}$$

and

$$\varrho_n(z) \equiv \varrho_{n,w}^{GK}(z) = \int_{-1}^{1}\frac{\pi_n(t)\pi^*_{n+1}(t)}{z-t}w(t)dt. \tag{33}$$

The integral representation (31) leads directly to the error estimate

$$|R_n(f)| \le \frac{l(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_n(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{34}$$

where $l(\Gamma)$ is the length of the contour Γ. To estimate the right-hand side, we study a magnitude of $|K_n(z)|$ on Γ. The knowledge of the maximum modulus of the kernel K_n is essential to obtain sharp error bounds.

Starting from the explicit expression of the kernel, we determine the location on the ellipses where the maximum modulus of the kernel is attained. We found sufficient conditions ensuring that there exists a ρ^* such that for each $\rho \geqslant \rho^*$ the kernel K_n attains its maximal absolute value at an intersection point of the ellipse with either the real or the imaginary axis, depending on the considered kernel. We derive the error bound (34) for quadrature (2) with respect to the weight function $w_i^R \quad (i = 1, 2, 3, 4)$ and quadrature (3) with respect to the weight function $w_i^L \quad (i = 1, 2)$.

3. Explicit expressions for the modulus of the kernel

The kernel is given by (32) and (33) for the quadrature (1). The expressions for the kernels related to quadrature (2) with respect to the wight function $w(w_1^R, w_2^R, w_3^R, w_4^R)$ respectively, and (3) with respect to the wight function $w(= w_1^L, w_2^L)$ are derived bellow.

3.1. Explicit kernel formula for Gauss-Radau extension and weight function $w = w_1^R$

From (10) and (16), we have

$$\pi_n(t)\pi_{n+1}^*(t) = \frac{1}{2^{2n+2}(1+t)^2}\Bigg\{ (T_{2n+3} + A_nT_{2n+2} + B_nT_{2n+1}) + A_nT_0 + T_1(1+B_n) + 2D_n(T_{n+2} + A_nT_{n+1} + B_nT_n) + C_n\sum_{k=0}^{n-1}(-1)^{k+1}B_n^{-k}\Big[(T_{2n+2-k} + A_nT_{2n+1-k} + B_nT_{2n-k}) + (T_{k+2} + A_nT_{k+1} + B_nT_k)\Big]\Bigg\}, \tag{35}$$

where T_n is the Chebyshev polynomial of the first kind and the following substitutions are introduced: $A_n = \frac{4(n+1)}{2n+1}$, $B_n = \frac{2n+3}{2n+1}$, $C_n = \frac{4(n+1)}{2n+3}$ and $D_n = (-1)^{n+1}\frac{2(n+1)}{2n+3}\left(\frac{2n+1}{2n+3}\right)^n$. Also, we applied substitutions $t = \cos\theta$, $T_0 = 1$, $T_1 = t$ and $T_nT_m = \frac{1}{2}(T_{n+m} + T_{|n-m|})$ according to (6).
Hence, introducing the following substitutions into (35)

$$\begin{aligned} F_n &= \int_{-1}^{1}\frac{T_n}{(1+t)^2(z-t)}\sqrt{\frac{(1+t)^3}{1-t}}dt = \int_{-1}^{1}\frac{T_n}{(z-t)}\frac{1}{\sqrt{1-t^2}}dt \\ &= \frac{\pi}{\sqrt{z^2-1}}(z-\sqrt{z^2-1})^n = \frac{2\pi}{(\xi-\xi^{-1})\xi^n}, \quad (n \geqslant 0) \end{aligned} \tag{36}$$

obtained according to [15, eq. 3.613.21], where $z = \frac{1}{2}(\xi + \xi^{-1}), |\xi| > 1, \xi = \rho e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi$, well known Joukowsky transform which maps the exterior of the unit circle into whole z-plane cut along $[-1, 1]$, and $T_j(\frac{1}{2}(\xi + \xi^{-1})) = \frac{1}{2}(\xi^j + \xi^{-j})$. Then we have

$$\begin{aligned} K_n(z) \equiv K_n(z;w) = &\frac{1}{2^{2n+2}}\frac{1}{\pi_n(z)\pi_{n+1}^*(z)}\Bigg\{ (F_{2n+3} + A_nF_{2n+2} + B_nF_{2n+1}) \\ &+ A_nF_0 + (1+B_n)F_1 + 2D_n(F_{n+2} + A_nF_{n+1} + B_nF_n) \\ &+ C_n\sum_{k=0}^{n-1}(-1)^{k+1}B_n^{-k}\Big(F_{2n+2-k} + A_nF_{2n+1-k} + B_nF_{2n-k} + F_{k+2} + A_nF_{k+1} + B_nF_k\Big)\Bigg\}, \end{aligned}$$

which gives us the explicit expression for $K_n(z)$ for quadrature (2) with respect to the weight function $w = w_1^R$:

$$K_n(z;w) = \frac{4\pi\left(1 + \frac{1}{2}\left(\xi + \xi^{-1}\right)\right)^2}{\xi - \xi^{-1}}\frac{J_N}{J_D},$$

where

$$
\begin{aligned}
J_N = {}& \xi^{-1}(1+B_n) + A_n + \left(1 + A_n\xi + B_n\xi^2\right) \\
&\cdot \left(\xi^{-(2n+3)} + 2D_n\xi^{-(n+2)} + C_n \sum_{k=0}^{n-1} (-1)^{k+1} B_n^{-k} \left(\xi^{-(k+2)} + \xi^{-(2n+2-k)} \right) \right)
\end{aligned}
$$

and

$$
\begin{aligned}
J_D = {}& (\xi + \xi^{-1})(1+B_n) + 2A_n + (1 + A_n\xi + B_n\xi^2) \\
&\cdot \left(\xi^{-(2n+3)} + 2D_n\xi^{-(n+2)} + C_n \sum_{k=0}^{n-1} (-1)^{k+1} B_n^{-k} \left(\xi^{-(k+2)} + \xi^{-(2n+2-k)} \right) \right) \\
&+ (\xi^2 + A_n\xi + B_n) \left(\xi^{2n+1} + 2D_n\xi^n + C_n \sum_{k=0}^{n-1} (-1)^{k+1} B_n^{-k} \left(\xi^k + \xi^{2n-k} \right) \right).
\end{aligned}
$$

3.2. *Explicit kernel formula for Gauss-Radau extension and weight function $w = w_2^R$*

In the case of quadrature (2) with respect to the weight function $w = w_2^R$, according to (11) and (17), we have

$$
\begin{aligned}
\pi_n(t)\pi_{n+1}^*(t) = {}& \frac{1}{2^{2n+3}(1+t)^2} \Bigg\{ U_{n+1}U_{n+2} + A_nU_{n+1}U_{n+1} + B_nU_nU_{n+1} \\
&+ \sum_{k=0}^{n} c_{n-k} \left(U_{n-k}U_{n+2} + A_nU_{n-k}U_{n+1} + B_nU_{n-k}U_n \right) \Bigg\} \\
= {}& \frac{1}{2^{2n+4}(1-t^2)(1+t)^2} \Bigg\{ (T_1 + A_nT_0 + B_nT_1) - (T_{2n+5} + A_nT_{2n+4} + B_nT_{2n+3}) \\
&+ \sum_{k=0}^{n} c_{n-k} \Big[(T_{k+2} + A_nT_{k+1} + B_nT_k) - (T_{2n-k+4} + A_nT_{2n-k+3} + B_nT_{2n-k+2}) \Big] \Bigg\}
\end{aligned} \tag{37}
$$

where U_n is the Chebyshev polynomial of the second kind and the substitutions introduced $A_n = \frac{4(n+3)}{2n+3}$, $B_n = \frac{(n+3)(2n+5)}{(n+1)(2n+3)}$, the rest is according (18), (19) and (20). Also, we applied equation (7) and earlier introduce F_n (36), we further calculate ϱ using

$$
\int_{-1}^{1} \frac{T_n}{z-t} \frac{(1-t)^{1/2}(1+t)^{5/2}}{(1-t^2)(1+t)^2} dt = \int_{-1}^{1} \frac{T_n}{z-t} \frac{1}{\sqrt{1-t^2}} dt = F_n,
$$

which is obviously equal to the previously introduced F_n. Hence,

$$
\begin{aligned}
\varrho_n(z) \equiv \varrho_{n,w}(z) = {}& \int_{-1}^{1} \frac{\pi_n(t)\pi_{n+1}^*(t)}{z-t} w(t)dt \\
= {}& \frac{1}{2^{2n+4}} \Bigg\{ (F_1 + A_nF_0 + B_nF_1) - (F_{2n+5} + A_nF_{2n+4} + B_nF_{2n+3}) \\
&+ \sum_{k=0}^{n} c_{n-k} \Big[(F_{k+2} + A_nF_{k+1} + B_nF_k) - (F_{2n-k+4} + A_nF_{2n-k+3} + B_nF_{2n-k+2}) \Big] \Bigg\},
\end{aligned}
$$

from which follows explicit kernel formulae

$$
K_n(z;w) = \frac{4\pi}{\xi - \xi^{-1}} \left(1 + \frac{1}{2}\left(\xi + \xi^{-1}\right) \right)^2 \left(1 - \left(\frac{1}{2}\left(\xi + \xi^{-1}\right) \right)^2 \right) \frac{J_N}{J_D},
$$

where

$$J_N = A_n + (1+B_n)\xi^{-1} - \left(1 + A_n\xi + B_n\xi^2\right)\left(\xi^{-(2n+5)} - \sum_{k=0}^{n} c_{n-k}\left(\xi^{-(k+2)} - \xi^{-(2n-k+4)}\right)\right),$$

and

$$\begin{aligned} J_D = & 2A_n + (1+B_n)\left(\xi + \xi^{-1}\right) - \left(\xi^2 + A_n\xi + B_n\right)\left(\xi^{2n+3} - \sum_{k=0}^{n} c_{n-k}\left(\xi^{k} - \xi^{2n-k+2}\right)\right) \\ & - \left(1 + A_n\xi + B_n\xi^2\right)\left(\xi^{-(2n+5)} - \sum_{k=0}^{n} c_{n-k}\left(\xi^{-(k+2)} - \xi^{-(2n-k+4)}\right)\right). \end{aligned}$$

3.3. Explicit kernel formula for Gauss-Radau extension and weight function $w = w_3^R$

For the quadrature (2) with respect to the weight $w = w_3^R$, we have

$$\begin{aligned} \pi_n(t)\pi_{n+1}^*(t) = & \frac{1}{2^{2n+3}(1+t)^2}\Bigg\{V_{n+2}V_{n+1} + A_nV_{n+1}V_{n+1} + B_nV_nV_{n+1} \\ & + \sum_{k=0}^{n} c_{n-k}\left(V_{n-k}V_{n+2} + A_nV_{n-k}V_{n+1} + B_nV_{n-k}V_n\right)\Bigg\} \\ = & \frac{1}{2^{2n+3}(1+t)^3}\Bigg\{(1+B_n)T_1 + A_nT_0 + T_{2n+4} + A_nT_{2n+3} + B_nT_{2n+2} \\ & + \sum_{k=0}^{n} c_{n-k}\left(\, T_{k+2} + A_nT_{k+1} + B_nT_k + T_{2n-k+3} + A_nT_{2n-k+2} + B_nT_{2n-k+1}\,\right)\Bigg\} \end{aligned}$$

where V_n is the Chebyshev polynomial of the third kind. The substitutions introduced are $A_n = \frac{2n+5}{n+1}$, $B_n = \frac{(n+2)(2n+5)}{(n+1)(2n+1)}$, the rest is according (22), (23) and (24). We applied substitutions (8) and (36) here, too, an integral obtained reduces to

$$\int_{-1}^{1} \frac{T_n}{z-t}\frac{(1-t)^{-1/2}(1+t)^{5/2}}{(1+t)^3}dt = \int_{-1}^{1} \frac{T_n}{z-t}\frac{1}{\sqrt{1-t^2}}dt = F_n.$$

Hence,

$$\begin{aligned} \varrho_n(z) \equiv \varrho_{n,w}(z) = & \int_{-1}^{1} \frac{\pi_n(t)\pi_{n+1}^*(t)}{z-t}w(t)dt \\ = & \frac{1}{2^{2n+3}}\Bigg\{A_nF_0 + (1+B_n)F_1 + F_{2n+4} + A_nF_{2n+3} + B_nF_{2n+2} \\ & + \sum_{k=0}^{n} c_{n-k}\left(\, F_{k+2} + A_nF_{k+1} + B_nF_k + F_{2n-k+3} + A_nF_{2n-k+2} + B_nF_{2n-k+1}\,\right)\Bigg\}. \end{aligned}$$

Then the explicit kernel formula is

$$K_n(z;w) = \frac{4\pi}{\xi - \xi^{-1}}\left(1 + \frac{1}{2}\left(\xi + \xi^{-1}\right)\right)^3 \frac{J_N}{J_D},$$

where

$$J_N = A_n + (1+B_n)\,\xi^{-1} + \left(1 + A_n\xi + B_n\xi^2\right)\left(\xi^{-(2n+4)} + \sum_{k=0}^{n} c_{n-k}\left(\xi^{-(2n-k+3)} + \xi^{-(k+2)}\right)\right),$$

and

$$J_D = 2A_n + (1+B_n)\left(\xi+\xi^{-1}\right) + \left(\xi^2 + A_n\xi + B_n\right)\left(\xi^{2n+2} + \sum_{k=0}^{n} c_{n-k}\left(\xi^k + \xi^{2n-k+1}\right)\right)$$
$$+ \left(1 + A_n\xi + B_n\xi^2\right)\left(\xi^{-(2n+4)} + \sum_{k=0}^{n} c_{n-k}\left(\xi^{-(k+2)} + \xi^{-(2n-k+3)}\right)\right).$$

3.4. Explicit kernel formula for Gauss-Radau extension and weight function $w = w_4^R$

Here

$$\pi_n(t)\pi^*_{n+1}(t) = \frac{1}{2^{2n+3}(1+t)^2(1-t)}\Big\{T_{2n+2} - T_{2n+4} + (A_n+B_n)(T_1 - T_{2n+3})$$
$$- C_n\sum_{k=1}^{n} B_n^{k-1}(T_{k+1} - T_{2n+3-k}) - A_nC_n\sum_{k=1}^{n} B_n^{k-1}(T_k - T_{2n-k+2})\Big\}$$

where the introduced substitution are: $A_n = \frac{n+2}{n+1}$, $B_n = -\frac{n+1}{n+2}$, $C_n = \frac{2n+3}{(n+2)^2}$, $t = \cos\theta$, $T_0 = 1$, $T_1 = t$ and (7). Then introducing substitutions (36), gives us the explicit expression for the *kernel*:

$$K_n(z;w) = \frac{4\pi\left(1+\frac{1}{2}\left(\xi+\xi^{-1}\right)\right)^2\left(1-\frac{1}{2}\left(\xi+\xi^{-1}\right)\right)}{\xi-\xi^{-1}}\frac{J_N}{J_D},$$

where

$$J_N = \xi^{-(2n+2)} - \xi^{-(2n+4)} + (A_n+B_n)\left(\xi^{-1} - \xi^{-(2n+3)}\right) - C_n\sum_{k=1}^{n} B_n^{k-1}\left(\xi^{-(k+1)} - \xi^{-(2n-k+3)}\right)$$
$$-A_nC_n\sum_{k=1}^{n} B_n^{k-1}\left(\xi^{-k} - \xi^{-(2n-k+2)}\right),$$

and

$$J_D = (A_n+B_n)\left(\xi+\xi^{-1}\right) + \xi^{2n+2} - \xi^{2n+4} + \xi^{-(2n+2)} - \xi^{-(2n+4)} - (A_n+B_n)\left(\xi^{2n+3} + \xi^{-(2n+3)}\right)$$
$$-C_n\sum_{k=1}^{n} B_n^{k-1}\left(\xi^{k+1} + \xi^{-(k+1)} - \xi^{-(2n+3-k)} - \xi^{2n+3-k}\right) - A_nC_n\sum_{k=1}^{n} B_n^{k-1}\left(\xi^k + \xi^{-k} - \xi^{2n-k+2} - \xi^{-(2n-k+2)}\right).$$

3.5. Explicit kernel formula for Gauss-Lobatto extension and weight function $w = w_1^L$

In the case of quadrature (3) with respect to the weight $w = w_1^L$ for n odd, following the same steps as in the previous cases, we have

$$\begin{aligned}\pi_n(t)\pi^*_{n+1}(t) &= \frac{1}{2^{2n+2}(t^2-1)}\Big\{U_{n+2}T_{n+1} - A_nU_nT_{n+1} + \sum_{k=1}^{(n-1)/2} A_n^{-k}(U_{n+2}T_{n-2k+1} - A_nU_nT_{n-2k+1})\\ &\quad + \frac{1}{2}A_n^{(n+1)/2}U_{n+2}T_0 - \frac{1}{2}A_nU_nT_0\Big\}\\ &= \frac{1}{2^{2n+3}(t^2-1)}\Big\{U_{2n+3} + U_1 - A_nU_{2n+1} + \sum_{k=1}^{(n-1)/2} A_n^{-k}(U_{2n-2k+3} + U_{2k+1})\\ &\quad - A_n\sum_{k=1}^{(n-1)/2} A_n^{-k}(U_{2n-2k+1} + U_{2k-1}) + A_n^{\frac{n+1}{2}}U_{n+2} - A_nU_n\Big\}\end{aligned} \tag{38}$$

were $A_n = \frac{n+3}{n+1}$, applied (9) and according to [15, eq. 3.613.3] or (see [1], cf. 2.25)

$$E_n = \int_{-1}^{1} \frac{U_n}{z-t}\frac{w(t)}{t^2-1}dt = \int_{-1}^{1} \frac{U_n}{z-t}\frac{(1-t^2)^{3/2}}{t^2-1}dt = -\int_{-1}^{1} \frac{U_n}{z-t}\sqrt{1-t^2}dt = -\frac{\pi}{\xi^{n+1}} \tag{39}$$

letting

$$U_n(z) = \frac{\xi^{n+1} - \xi^{-(n+1)}}{\xi - \xi^{-1}}$$

finally

$$K_n(z;w) = -\pi\left(\left(\frac{1}{2}\left(\xi + \xi^{-1}\right)\right)^2 - 1\right)\left(\xi - \xi^{-1}\right)\frac{J_N}{J_D}$$

where numerator J_N and denominator J_D are

$$J_N = \xi^{-2} + \xi^{-(n+3)}(A_n^{(n+1)/2} - A_n\xi^2) + (1 - A_n\xi^2)\left(\xi^{-(2n+4)} + \sum_{k=1}^{(n-1)/2} A_n^{-k}(\xi^{-(2n-2k+4)} + \xi^{-(2k+2)})\right)$$

$$\begin{aligned} J_D = &(\xi^2 - A_n)\left(\xi^{2n+2} + \sum_{k=1}^{(n-1)/2} A_n^{-k}(\xi^{2k+2} + \xi^{2n+2-2k})\right) - (1 - A_n\xi^2)\left(\xi^{-(2n+4)} + \sum_{k=1}^{(n-1)/2} A_n^{-k}(\xi^{-(2n-2k+4)} + \xi^{-(2k+2)})\right) \\ &+ A_n^{(n+1)/2}(\xi^{n+3} - \xi^{-(n+3)}) - A_n(\xi^{n+1} - \xi^{-(n+1)}) \end{aligned}$$

For n even we have

$$\pi_n(t)\pi_{n+1}^*(t) = \frac{1}{2^{2n+3}(t^2-1)}\left\{U_{2n+3} + U_1 - A_nU_{2n+1} + \sum_{k=1}^{(n-1)/2} A_n^{-k}\left[(U_{2n-2k+3} + U_{2k+1}) - A_n\left(U_{2n-2k+1} + U_{2k-1}\right)\right]\right\} \tag{40}$$

Finally,

$$K_n(z;w) = \frac{-\pi}{\xi^{2n+4}}\left(\left(\frac{1}{2}\left(\xi + \xi^{-1}\right)\right)^2 - 1\right)\left(\xi - \xi^{-1}\right)\frac{J_N}{J_D}$$

where

$$J_N = \xi^{2n+2} + \left(1 - A_n\xi^2\right)\left(1 + \sum_{k=1}^{n/2} A_n^{-k}\left(\xi^{2k} + \xi^{2n-2k+2}\right)\right),$$

$$J_D = \xi^2 - \xi^{-2} + \left(\xi^2 - A_n\right)\left(\xi^{2n+2} + \sum_{k=1}^{n/2} A_n^{-k}(\xi^{2k-1} + \xi^{2n-2k+1})\right) - (1 + A_n\xi^2)\left(\xi^{-(2n+4)} + \sum_{k=1}^{n/2} A_n^{-k}(\xi^{-(2n-2k+3)} + \xi^{-(2k+1)})\right).$$

3.6. Explicit kernel formula for Gauss-Lobatto extension and weight function $w = w_2^L$

For the quadrature (3) and weight function $w = w_2^L$, consider (29) and (15), we have

$$
\begin{aligned}
\pi_n(t)\pi_{n+1}^*(t) = & \frac{1}{2^{2n+5}(1-t^2)^2}\Bigg\{U_{n+1}U_{n+4} + A_nU_{n+1}U_{n+2} + B_nU_{n+1}U_n \\
& + \sum_{k=1}^{\lfloor n/2\rfloor} c_{n-2k+1}\left(U_{n-2k+1}U_{n+4} + A_nU_{n-2k+1}U_{n+2} + B_nU_{n-2k+1}U_n\right)\Bigg\} \\
& + \frac{C_n}{2^{n+4}(1-t^2)^2}\Big\{U_{n+4} + A_nU_{n+2} + B_nU_n\Big\} \\
= & \frac{1}{2^{2n+6}(1-t^2)^3}\Bigg\{T_3 + (A_n+B_n)T_1 - T_{2n+7} - A_nT_{2n+5} - B_nT_{2n+3} \\
& + \sum_{k=1}^{\lfloor n/2\rfloor} c_{n-2k+1}\left(T_{2k+3} + A_nT_{2k+1} + B_nT_{2k-1} - T_{2n-2k+7} - A_nT_{2n-2k+5} - B_nT_{2n-2k+3}\right)\Bigg\} \\
& + \frac{C_n}{2^{n+4}(1-t^2)^2}\Big\{U_{n+4} + A_nU_{n+2} + B_nU_n\Big\}
\end{aligned}
\tag{41}
$$

where the following substitutions are introduced: $A_n = \frac{-2(n+5)}{n+2}$, $B_n = \frac{(n+4)(n+5)}{(n+1)(n+2)}$ and $C_n = \frac{1+(-1)^{n-1}}{4}c_0U_0(t)$. Also, we applied substitutions (7), (39) and (36), where

$$
E_{1n} \;=\; \int_{-1}^{1}\frac{U_n}{z-t}\frac{w(t)}{t^2-1}dt = \int_{-1}^{1}\frac{U_n}{z-t}\frac{(1-t^2)^{5/2}}{(1-t^2)^2}dt = \int_{-1}^{1}\frac{U_n}{z-t}\sqrt{1-t^2}dt = \frac{\pi}{\xi^{n+1}} = -E_n. \tag{42}
$$

Explicit expression for the kernel is

$$
K_n(z;w) = \frac{J_N}{J_D},
$$

where

$$
\begin{aligned}
J_N = & \frac{\pi}{2^{2n+5}(\xi-\xi^{-1})}\Bigg\{\xi^{-3} + (A_n+B_n)\xi^{-1} - (1 + A_n\xi^2 + B_n\xi^4)\Bigg(\xi^{-(2n+7)} - \sum_{k=1}^{\lfloor n/2\rfloor} c_{n-2k+1}(\xi^{-(2k+3)} - \xi^{-(2n-2k+7)})\Bigg)\Bigg\} \\
& + \frac{\pi C_n}{2^{n+4}\xi^{n+5}}\left(1 + A_n\xi^2 + B_n\xi^4\right),
\end{aligned}
$$

and

$$
\begin{aligned}
J_D = & \frac{1}{2^{2n+7}(1-t^2)^3}\Bigg\{(A_n+B_n)(\xi+\xi^{-1}) + (\xi^3+\xi^{-3}) \\
& - (\xi^4 + A_n\xi^2 + B_n)\Bigg(\xi^{2n+3} - \sum_{k=1}^{\lfloor n/2\rfloor} c_{n-k2+1}(\xi^{2k-1} - \xi^{2n+3-2k})\Bigg) \\
& - (1 + A_n\xi^2 + B_n\xi^4)\Bigg(\xi^{-(2n+7)} - \sum_{k=1}^{\lfloor n/2\rfloor} c_{n-k2+1}(\xi^{-(2k+3)} - \xi^{-(2n+7-2k)})\Bigg)\Bigg\} \\
& + \frac{C_n}{2^{n+4}(1-t^2)(\xi-\xi^{-1})}\Big\{\xi^{n+1}\left(\xi^4 + A_n\xi^2 + B_n\right) + \xi^{-(n+5)}\left(1 + A_n\xi^2 + B_n\xi^4\right)\Big\}
\end{aligned}
$$

4. Maximum of modulus of the kernels

To derive an upper bound for the modulus of the kernel, using the obtained expressions in the previous section, we formulated the following two theorems and proved that we can completely describe the maximum of the modulus of the kernel for large enough values of ρ. In these theorems, we investigate asymptotic by inspecting the leading coefficients only.

Theorem 4.1. *Let A and B be the real numbers different each other and* $\xi = \rho e^{i\varphi}$ *and*

$$I(\varphi) = \frac{\xi + A + o\left(\frac{1}{\rho}\right)}{\xi + B + o\left(\frac{1}{\rho}\right)}, \quad (\rho \to \infty)$$

if $A < B$, *then there exists* ρ^* *such that for each* $\rho > \rho^*$, $\max_{\varphi\in[0,2\pi)} \left|I(\varphi)\right| = |I(\pi)|$ *and*
if $A > B$, *then there exists* ρ^* *such that for each* $\rho > \rho^*$, $\max_{\varphi\in[0,2\pi)} \left|I(\varphi)\right| = |I(0)|$.

Theorem 4.2. *Let A and B be the real numbers different each other and* $\xi = \rho e^{i\varphi}$ *and*

$$I(\varphi) = \frac{\xi^2 + A + o\left(\frac{1}{\rho}\right)}{\xi^2 + B + o\left(\frac{1}{\rho}\right)}, \quad (\rho \to \infty)$$

if $A < B$, *then there exists* ρ^* *such that for each* $\rho > \rho^*$, $\max_{\varphi\in[0,2\pi)} \left|I(\varphi)\right| = \left|I\left(\frac{\pi}{2}\right)\right|$
and if $A > B$, *then there exists* ρ^* *such that for each* $\rho > \rho^*$, $\max_{\varphi\in[0,2\pi)} \left|I(\varphi)\right| = |I(0)| = |I(\pi)|$.

The proofs of the theorems are a direct consequence of ([11, Lemma 4.1 and Theorem 4.1]).

We found sufficient conditions for the modulus of the kernel to be maximized on one of the axes for each ρ large enough. The theorems confirm that for given n the argument θ of the maximum point z of the $|Kn(z)|$ on ellipses eventually stabilizes at some value $\theta \in \{0, \pi/2, \pi\}$ as ρ increases. The smallest ρ for which this happens was denoted by ρ^*. We confirmed that the theorems can be directly applied, to each of the six calculated kernels, empirically using MATLAB where the leading coefficients are extracted and checked against theorems. Empirical experiments supported propositions. Numerical experiments showed that the corresponding value of ρ^* stabilizes close to 1.1 in some examples, but in all cases error bound is satisfactory. Due to the complexity of the calculated kernel expressions determining ρ makes sense only empirically. We also encountered situation where the asymptotic behavior of the kernel is inconsistent with the propositions. However, it only happens for $\rho \geqslant 10^8$, where the values of the function $|Kn(z)|$ are so small that the difference between max and min value is negligible (see Figure 2).

Changes in the point of reaching the maximum of the modulus of the kernels refer to $n = 4, 6, 8, 10$, the most common n in our experiments, for (2) and (3) are presented in the figures 1. The behavior of the kernel depends on n.

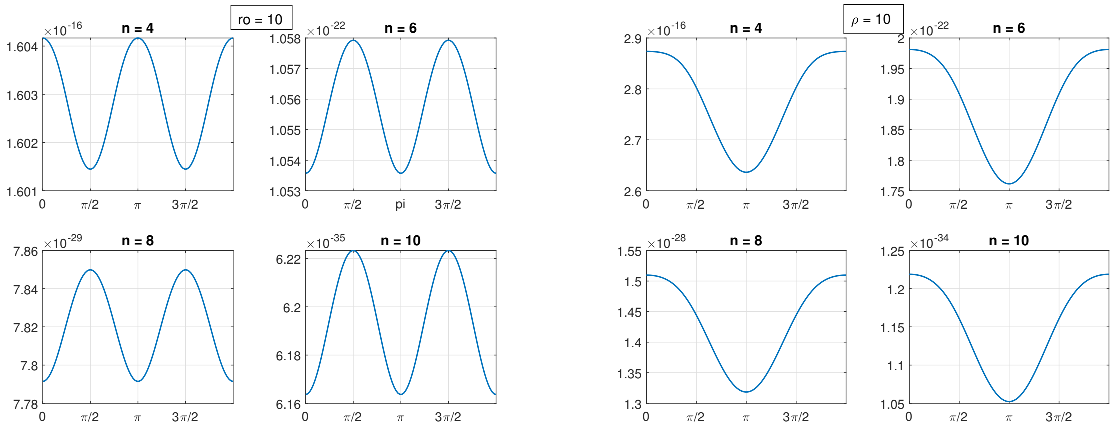

Figure 1: Graph of the modulus of the kernel: left for Gauss-Radau q.f. (2) and weight function w_4^R; right: Gauss-Lobatto q.f. (3) with respect to the weight function w_1^L when n is even.

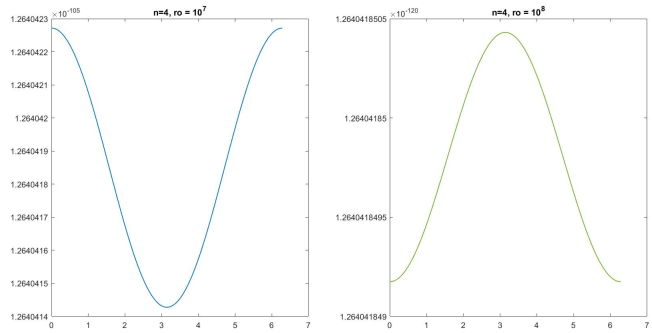

Figure 2: Graph of the modulus of the kernel for Gauss-Radau q.f. (2) and weight function w_4^R for very large ρ.

4.1. *Numerical results*

The length of the ellipse ε_p can be estimated by (see [13, eq.(2.2)])

$$l(\varepsilon_p) \leq 2\pi a_1\Big(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\Big), \quad a_1 = \frac{\rho + \rho^{-1}}{2}.$$

Therefore, we propose for use the error bound $|R_n(f)| \leqslant r_n(f)$, where

$$r_n(f) = \inf_{\rho_n^* < \rho < \rho_{max}} \Big[a_1\Big((1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\Big) \cdot \Big(\max_{z\in\varepsilon_p} |K_n(z)|\Big)\Big(\max_{z\in\varepsilon_p} |f(z)|\Big)\Big].$$

The aim here is to estimate the minimal value of ρ^* such that K_n attains the maximum value on ε_p, with the respect to the domain of ρ: $(1, +\infty)$. In order to check the proposed error bounds, we made several tests and compared them with respect to the exact (actual) errors. Examples are made for some special functions, appearing in the literature. In the case of the function $f_1(t) = e^{\omega t^2}$ it holds that $\max_{z\in\varepsilon_p} |f_1(z)| = e^{\omega a_1^2}$ $\Big(a_1 = \frac{1}{2}(\rho + \rho^{-1})\Big)$, $\rho_{max} = +\infty$.

In the case of function $f_2(t) = e^{\cos \omega t}$, where $\omega > 0$ holds $\max_{z\in\varepsilon_p} |f_2(z)| = |e^{\cosh(\omega b_1)}|$ $\Big(b_1 = \frac{1}{2}\Big(\rho - \rho^{-1}\Big)\Big)$.

The obtained results are shown in the tables 1 - 7.

Table 1: Error bounds and minimal ρ from the the theorem 4.1 for Gauss-Radu formulae (2) and weight function w_1^R.

(n, ω)	ρ^*	r_n	Error	$I = I(f_1) = \int_{-1}^{1} \frac{e^{\omega t^2} dt}{\sqrt{1-t^2}}$
(4,16)	1.6	3.59 (+06)	1.12 (+05)	4.004...(+6)
(4,8)	2.1	6.42 (+01)	1.89 (+00)	1.938...(+3)
(4,4)	2.8	2.67 (-03)	6.22 (-04)	5.291...(+1)
(4,2)	3.9	4.69 (-05)	7.54 (-07)	1.081...(+1)
(4,1)	5.4	1.64 (-07)	1.67 (-09)	5.508...(+0)
(4,0.5)	7.6	8.17 (-10)	4.93 (-12)	4.097...(+0)
(6,8)	2.4	4.08 (-01)	8.38 (-03)	1.938...(+3)
(8,8)	2.7	1.24 (-03)	1.99 (-05)	1.938...(+3)
(10,8)	2.9	2.12 (-06)	2.83 (-08)	1.938...(+3)
(12,8)	3.2	2.29 (-09)	2.57 (-11)	1.938...(+3)
(14,8)	3.4	1.61 (-12)	1.93 (-13)	1.938...(+3)

Table 2: Error bounds and minimal ρ from the theorem 4.1 for Gauss-Radu formulae (2) and weight function w_1^R.

(n, ω)	ρ^*	r_n	Error	$I = I(f_2) = \int_{-1}^{1} \frac{e^{\cos \omega t} dt}{\sqrt{1-t^2}}$
(6,16)	1.11	9.92 (+01)	3.11 (-01)	3.459...(+0)
(6,8)	1.26	4.03 (+00)	2.13 (-02)	4.432...(+3)
(6,2)	2.91	5.65 (-07)	5.05 (-09)	4.457...(+0)
(6,1)	5.53	1.63 (-12)	8.31 (-15)	6.842...(+0)
(12,8)	1.33	2.80 (-03)	2.44 (-05)	4.432...(+0)
(20,8)	1.41	9.02 (-06)	1.05 (-08)	4.432...(+0)
(30,8)	1.46	1.18 (-11)	1.36 (-13)	4.432...(+0)

Table 3: Error bounds and minimal ρ from the theorem 4.1 for Gauss-Radu formulae (2) and weight function w_4^R.

(n, ω)	ρ^*	r_n	Error	$I = I(f_1) = \int_{-1}^{1} \frac{e^{\omega t^2} dt}{\sqrt{1-t^2}}$
(4,16)	1.6	2.37 (+05)	1.12 (+05)	4.004...(+6)
(4,8)	2.0	3.95 (+00)	1.89 (+00)	1.938...(+3)
(4,4)	2.7	1.98 (-03)	6.22 (-04)	5.291...(+1)
(4,2)	3.8	4.53 (-06)	7.54 (-07)	1.081...(+1)
(4,1)	5.3	1.99 (-08)	1.67 (-09)	5.508...(+0)
(4,0.5)	7.5	1.18 (-10)	4.93 (-12)	4.097...(+0)
(6,8)	2.3	2.26 (-02)	8.38 (-03)	1.938...(+3)
(8,8)	2.6	7.94 (-05)	1.99 (-05)	1.938...(+3)
(10,8)	2.9	1.56 (-07)	2.83 (-08)	1.938...(+3)
(12,8)	3.1	1.84 (-10)	2.57 (-11)	1.938...(+3)
(14,8)	3.3	4.12 (-13)	1.93 (-13)	1.938...(+3)

Table 4: Error bounds and minimal ρ from the theorem 4.1 for Gauss-Radau extension (2) and weight function w_4^R.

(n,ω)	ρ^*	r_n	Error	$I = I(f_2) = \int_{-1}^{1} \frac{e^{\cos(\omega t)}dt}{\sqrt{1-t^2}}$
(10,16)	1.11	8.37 (-01)	5.61 (-02)	3.459...(+0)
(14,16)	1.11	1.55 (-01)	7.67 (-03)	3.459...(+0)
(20,16)	1.11	1.34 (-02)	1.03 (-04)	3.459...(+0)
(30,16)	1.21	7.02 (-05)	1.29 (-06)	3.459...(+0)
(40,16)	1.21	1.53 (-07)	3.76 (-09)	3.459...(+0)
(50,16)	1.21	3.95 (-10)	1.15 (-11)	3.459...(+0)
(60,16)	1.21	1.08 (-12)	3.18 (-14)	3.459...(+0)
(8,8)	1.31	3.20 (-02)	3.36 (-04)	4.432...(+0)
(12,8)	1.31	5.97 (-04)	2.44 (-05)	4.432...(+0)
(20,8)	1.41	1.36 (-07)	1.05 (-08)	4.432...(+0)
(30,8)	1.41	2.99 (-12)	1.36 (-13)	4.432...(+0)
(32,8)	1.41	2.88 (-13)	1.31 (-14)	4.432...(+0)

Table 5: The values of error bounds $r_n(f_1, w_1^L)$, actual error and smallest possible ρ from the theorem 4.2.

(n,ω)	ρ^*	r_n	Error	$I = I(f_1) = \int_{-1}^{1} \frac{e^{\omega t^2}dt}{\sqrt{1-t^2}}$
(4,16)	1.4	1.60 (+05)	1.12 (+05)	4.004...(+6)
(4,2)	3.8	2.37 (-06)	7.54 (-07)	1.081...(+1)
(4,1)	5.3	2.12 (-08)	1.67 (-09)	5.508...(+0)
(4,0.5)	7.5	6.50 (-11)	4.93 (-12)	4.097...(+0)
(6,8)	2.3	1.02 (-02)	8.38 (-03)	1.938...(+3)
(8,8)	2.6	3.63 (-05)	1.99 (-05)	1.938...(+3)
(10,8)	2.9	7.24 (-08)	2.83 (-08)	1.938...(+3)
(12,8)	3.1	8.56 (-11)	2.57 (-11)	1.938...(+3)
(14,8)	3.3	6.62 (-13)	1.93 (-13)	1.938...(+3)

Table 6: The values of error bounds $r_n(f_2, w_1^L)$, actual error and smallest possible ρ from the theorem 4.2

(n,ω)	ρ^*	r_n	Error	$I = I(f_2) = \int_{-1}^{1} \frac{e^{\cos(\omega t)}dt}{\sqrt{1-t^2}}$
(6,16)	1.11	2.48 (+00)	3.11 (-01)	3.459...(+0)
(6,8)	1.27	9.87 (-02)	2.13 (-02)	4.432...(+3)
(6,4)	1.72	1.93 (-04)	8.69 (-05)	2.717...(+0)
(6,2)	2.88	1.97 (-08)	5.05 (-09)	4.457...(+0)
(6,1)	5.49	1.08 (-13)	8.31 (-15)	6.842...(+0)
(12,8)	1.34	2.05 (-04)	2.44 (-05)	4.432...(+0)
(20,8)	1.31	5.09 (-08)	1.05 (-08)	4.432...(+0)

Table 7: The values of error bounds $r_n(f_2, w_2^L)$, actual error and smallest possible ρ from the theorem 4.2

(n,ω)	ρ^*	r_n	Error	$I = I(f_2) = \int_{-1}^{1} \frac{e^{\cos(\omega t)}}{\sqrt{1-t^2}}$
(6,16)	1.11	1.38 (+00)	3.11 (-01)	3.459...(+0)
(6,8)	1.21	1.32 (-01)	2.13 (-02)	4.432...(+3)
(6,4)	1.61	3.40 (-03)	8.69 (-05)	2.717...(+0)
(10,16)	1.11	3.10 (-01)	5.61 (-02)	3.459...(+0)
(14,16)	1.11	1.20 (-01)	7.67 (-03)	3.459...(+0)
(20,16)	1.11	3.00 (-02)	1.03 (-04)	3.459...(+0)
(8,8)	1.21	3.91 (-02)	3.36 (-04)	4.432...(+0)
(12,8)	1.21	5.80 (-03)	2.44 (-05)	4.432...(+0)

5. Conclusion

We got the effective error bounds for Gauss-Radau (2) and Gauss-Lobatto (3) q.f. when the integrand is a function analytic inside of a region bounded by confocal ellipses that contain the interval of integration. We established the minimal value for ρ (ρ^*), which, in addition to the proved theorems, completely describes the behavior of the kernels. Numerical results confirmed that error bounds obtained in this paper are close to actual.

References

[1] W.Gautschi, S.Notaris *Gauss-Kronrod quadrature formulae for weight functions of Bernstein-Szego type.* Journal of Computational and Applied Mathematics, **25** (1989), 199–224.
[2] W. Gautschi, S.Li, *Gauss-Radau and Gauss-Lobatto quadratures with double end points*, Journal of Computational and Applied Mathematics, **34** (1991), (343–36).
[3] W. Gautschi, *On the remainder term for analytic functions of Gauss-Lobatto and Gauss-Radau quadratures*, Rocky Mountain J. Math., **21** (1991), (209–226).
[4] W. Gautschi and S. E. Notaris, *Gauss-Kronrod quadrature formulas for weight function of Bernstein-Szegő type*, J. Comput. Appl. Math., **25** (1989), (199–224); erratum in J. Comput. Appl. Math., **27** (1989), (429).
[5] W. Gautschi and R. S. Varga, *Error bounds for Gaussian quadrature of analytic functions*, SIAM J. Numer. Anal., **20** (1983), (1170–1186).
[6] W. Gautschi, E. Tychopoulos, and R. S. Varga, *A note on the contour integral representation of the remainder term for a Gauss-Chebyshev quadrature rule*, SIAM J. Numer. Anal., **27** (1990), (219–224).
[7] S. Li, *Kronrod extension of generalized Gauss-Radau and Gauss-Lobatto formulae*, Journal of Mathematics, **26** (1996), (1455–1472).
[8] Lj. V. Mihić, A. V. Pejčev, and M. M. Spalević, *Error bounds for Gauss-Lobatto quadrature formula with multiple end points with Chebyshev weight function of the third and the fourth kind*, Filomat, **30:1** (2016), (231–239).
[9] Lj. V. Mihić, A. V. Pejčev, and M. M. Spalević, *Error estimations of Turán formulas with Gori-Micchelli and generalized Chebyshev weight functions*, Filomat, **32:20** (2018), (6927–6936).
[10] D. LJ. Đukić, A. V. Pejčev, and M. M. Spalević, *The error bounds of Gauss-Kronrod quadrature formulae for weight functions of Bernstein-Szego type*, Numer Algor, **77** (2018), (1003–1028).
[11] D. LJ. Đukić, R. M. Mutavdžić Đukić, A. V. Pejčev, and M. M. Spalević, *Error estimates of gaussian-type quadrature formulae for analytic functions on ellipses—a survey of recent results.*, Electron. Trans. Numer. Anal. 53 (2020), pp. 352–382.
[12] S. E. Notaris, *Integral formulas for Chebyshev polynomials and the error term of interpolatory quadrature formulas for analytic functions*, Math. Comp., **75** (2006), (1217–1231).
[13] Scherer, R., Schira, T., *Estimating quadrature errors for analytic functions using kernel representations and biorthogonal systems*, Numer. Math. **84** (2000), (497–518).
[14] T. Schira, *The remainder term for analytic functions of Gauss-Lobatto quadratures*, J. Comput. Appl. Math., **76** (1996), (171–193).
[15] I.S. Gradshteyn, I.M. Ryzhik , *Table of integrals, series, and products*, Academic Press, New York (1965).