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The Results on Coincidence and Common Fixed Points for a New Type Multivalued Mappings in *b*-Metric Spaces

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Abstract: In this paper, we obtain the results of coincidence and common fixed points in *b*-metric spaces. We work with a new type of multivalued quasi-contractive mapping with nonlinear comparison functions. Our results generalize and improve several recent results. Additionally, we give an application of the obtained results to dynamical systems.

Keywords: coincidence fixed; common fixed point; nonlinear quasi-contractions; *b*-metric spaces

MSC: 54H25; 54E99; 47H10

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1. Introduction

The first result of a fixed point for quasi-contractive mappings was presented by Lj. Ćirić [1] in 1974. The result of Ćirić is the most general result with linear comparison function in metric fixed-point theory (see [2,3]).

Existence and uniqueness of fixed point for quasi-contractive mapping with nonlinear comparison function on metric spaces, considered by J. Daneš [4], A. A. Ivanov [5], I. Aranđelović, M. Rajović and V. Kilibarda [6] and M. Bessenyei [7].

In the paper [8], S. Aleksić et al. proved a fixed-point theorem for quasi-contractive mappings defined by linear quasi-contractive conditions on *b*-metric spaces.

The results of common fixed points as a generalization result of Ćirić was obtained in [9] and in [10] with linear and nonlinear comparison functions, respectively.

The result of Ćirić [1] was generalized for multivalued quasi-contractive mappings by B. Fisher [11] (for linear cases) and by Ćirić and Ume [12] (for nonlinear cases). Further extension of these results was presented in [13].

A common fixed-point result for single-valued nonlinear quasi-contractions was presented by Z. D. Mitrović et al [14].

The notion of common fixed points for a hybrid pair of single-valued and multivalued mappings was introduced by S. L. Sigh, K. S. Ha, and Y. J. Cho [15]. Further important common fixed-point results for mappings of this type were presented in [16–18]. Theorems for the existence of a solution on Voltera integral inclusion and certain systems of functional equations arising in dynamic programming can be found in [19].

In this article, we give the results on the existence of a point of coincidence and a common strict fixed point for a hybrid pair of single-valued and multivalued mappings defined in *b*-metric spaces, which satisfy quasi-contractive inequality with nonlinear comparison function. Our results extend and generalize the results presented in [1,4–6,11–14]. Finally, we give an application of our results in the topological theory of set-valued dynamical systems.

2. Preliminaries

The concept of b-metric space was introduced by I. A. Bakhtin [20] and S. Czerwik [21].

Definition 1. *Let* X *be a nonempty set,* $d : \mathrm{X} \times \mathrm{X} \to [0, +\infty)$ *and* $s \in [0, +\infty)$ *such that for all* $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$*:*

(1_b) $d(\mathrm{x}, \mathrm{y}) = 0$ *if and only if* $\mathrm{x} = \mathrm{y}$*;*
(2_b) $d(\mathrm{x}, \mathrm{y}) = d(\mathrm{y}, \mathrm{x})$*;*
(3_b) $d(\mathrm{x}, \mathrm{z}) \leq s[d(\mathrm{x}, \mathrm{y}) + d(\mathrm{y}, \mathrm{z})]$.

A triplet (X, d, s) *is called a b-metric space.*

Some results in b-metric spaces in the last ten years can be seen in [22–38].

A b-metric space can is a topological space with the topology induced with family sets $\{B_n(x) : n = 1, 2, \ldots\}$ as a base of neighborhood filter of the point $x \in X$. The ball $\{B_n(x)\}$ is defined by

$$B_n(x) = \{y \in X : d(x, y) < \frac{1}{n}\}.$$

We call the sequence $(x_n) \subseteq X$ Cauchy if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that it holds $d(x_n, x_m) < \epsilon$ for all $m, n \geq n_0$.

A b-metric space (X, d, s) is complete if each Cauchy sequence converges.

Let 2^X and $\mathcal{B}(X)$ denote the family of all nonempty sets and all nonempty bounded subsets of X, respectively.

The function $\delta : \mathcal{B}(X) \times \mathcal{B}(X) \to [0, +\infty)$ defined by

$$\delta(P, Q) = \sup\{d(p, q) : p \in P, q \in Q\},$$

for any $P, Q \in \mathcal{B}(X)$.

The $\delta(\{p\}, \{q\})$, $\delta(\{p\}, Q)$ and $\delta(P, \{q\})$ we denote by $\delta(p, q)$, $\delta(p, Q)$ and $\delta(P, q)$ respectively.

From definition of δ, it follows that:

(1) $\delta(P, Q) \geq 0$;
(2) $\delta(P, Q) = \delta(Q, P)$;
(3) $\delta(P, R) \leq s[\delta(P, Q) + \delta(Q, R)]$,

for any $P, Q, R \in \mathcal{B}(X)$.

The $\mathrm{diam}(P)$ denotes the diameter of $P \subseteq X$, defined by

$$\mathrm{diam}(P) = \delta(P, P).$$

Let X and Y be nonempty sets and $F : X \to 2^Y$, $g : X \to Y$. If for some $x \in X$ there exists $y \in Fx$ such that $y = g(x)$, then x is a coincidence point of the multivalued function F and the single-valued function g and y is their point of coincidence.

An element $x_0 \in X$ is a fixed point of map F if $x_0 \in F(x_0)$.

If $F(x_0) = \{x_0\}$ then x_0 is a strict fixed point (or a stationary point) of F.

Definition 2. *Let* X *be an arbitrary set,* $F : X \to 2^X$ *and* $g : X \to X$*. The hybrid pair of multivalued function* F *and single-valued function* g *commute in* $x \in X$ *if* $F(gx) = g(Fx)$*. The* F *and* g *are weakly compatible if* F *and* g *commute at their coincidence points.*

Lemma 1. *Let* X *be an arbitrary set,* $F : X \to 2^X$ *and* $g : X \to X$ *be hybrid pair of weakly compatible functions. If* F *and* g *have a unique point of coincidence* $z \in X$ *i.e., there exists* $x \in X$ *such that* $z \in Fx$ *and* $gx = z$ *then* z *is unique common fixed point of* F *and* g*.*

Proof. Suppose that there exists unique $z \in X$ such that $z \in Fx$ and $gx = z$, for some $x \in X$. Then $Fz = g(Fx)$ because F and g commute at x. This implies that $g(z) \in F(z)$. Therefore, $z = gz$ because z is unique point of coincidence. □

Let $I \subseteq \mathbb{R}$ be an open interval. Function $\varphi : I \to \mathbb{R}$ is upper semicontinuous if

$$\overline{\lim}_{t\to r}\varphi(t) < r,$$

for any $r \in I$. By Φ we denote the family functions $\phi : [0,+\infty) \to [0,+\infty)$ such that:
(i) $\phi(0) = 0$;
(ii) $\phi(t) < t$ for all $t > 0$;
(iii) $\lim_{t\to+\infty}(t - \phi(t)) = +\infty$.
Let

$$\Phi_1 = \{\phi \in \Phi : \overline{\lim}_{u\to r+}\phi(u) < r \text{ for } r > 0 \text{ and } \varphi \text{ is monotone nondecreasing}\},$$

$$\Phi_2 = \{\phi \in \Phi : \overline{\lim}_{t\to r}\phi(t) < r \text{ for any } r > 0\}.$$

We have $\Phi_1 \subseteq \Phi_2$. Indeed, if φ is monotone nondecreasing, then $\overline{\lim}_{t\to r+}\varphi(t) \leq \varphi(r) < r$.
The following lemmas were proved in [6].

Lemma 2 ([6])**.** *If $\phi \in \Phi_2$ then there exists $\theta \in \Phi_1$ such that*

$$\phi(t) \leq \theta(t) < t,$$

for each $t > 0$.

Lemma 3 ([6])**.** *Let $\varphi_1, \dots, \varphi_n \in \Phi_1$. Then there exists $\psi \in \Phi_1$ such that*

$$\varphi_k(x) \leq \psi(x) < x,$$

for all $k \in \{1, \dots, n\}$ and $x > 0$.

We also need the following result, proved by J. Jachymski and I. Jóźwik [39].

Lemma 4 ([39])**.** *Assume that $\psi \in \Phi$ is upper semicontinuous function. Then there exists a continuous and nondecreasing function $\varphi \in \Phi$ such that $\psi(t) \leq \varphi(t)$ for all $t > 0$.*

Remark 1. *By Lemma 4, we learn that $\varphi \in \Phi_1$ because φ is continuous.*

3. Main Results

In this section, we consider a new type of multivalued quasi-contractive mapping with nonlinear comparison functions. We first give the following definition.

Definition 3. *Let X and Y be arbitrary sets, $g : X \to Y$ be single-valued and $F : X \to 2^Y$ be a multivalued function such that $F(X) \subseteq g(X)$ and $(x_n) \subseteq X$ sequence such that $gx_{i+1} \in Fx_i$. Then sequence $(y_n) \subseteq Y$ defined by $y_i = gx_i$ is called a Jungck sequence of hybrid pair F and g with an initial point x_0.*

Now we present our main result.

Theorem 1. *Let X be a nonempty set and (Y, d, s) be a b-metric space. Let $F : X \to \mathcal{B}(Y)$ be multivalued function and $g : X \to Y$ a single-valued function. Suppose that $F(X) \subseteq g(X)$ and that $g(X)$ is a complete subspace of Y. If there exist the functions $\varphi_i : [0,+\infty) \to [0,+\infty)$, such that $s\varphi_i \in \Phi_2$, $i = 1,2,3,4,5$ and*

$$\delta(Fx, Fy) \leq \max\{\varphi_1(\delta(gx, gy)), \varphi_2(\delta(gx, Fx)), \varphi_3(\delta(gy, Fy)), \varphi_4(\delta(gx, Fy)), \varphi_5(\delta(Fx, gy)\}, \tag{1}$$

for any $x, y \in X$, then there exists the unique point of coincidence $z \in Y$ of F and g and $z = \lim_{n \to +\infty} y_n$, where (y_n) is Jungck sequence defined by F and G.

Additionally, if F and g are weakly compatible and $X = Y$, then z is the unique common strict fixed point of F and g.

Proof. From Lemma 2, it follows that there exist functions $\varphi_k^* : [0, +\infty) \to [0, +\infty)$ such that $s\varphi_k^* \in \Phi_1$ and

$$\varphi_k(t) \leq \varphi_k^*(t) < t,$$

for each $t > 0$, $k = 1, 2, 3, 4, 5$. From Lemma 3 follows that there exists a real function $\varphi_0 : [0, +\infty) \to [0, +\infty)$ such that $s\varphi_0 \in \Phi_1$ and

$$\varphi_k^*(t) \leq \varphi_0(t) < \frac{t}{s}, k = 1, 2, 3, 4, 5, \text{ for each } t > 0.$$

Hence, for every $x, y \in X$ we obtain

$$\delta(Fx, Fy) \leq \max\{\varphi_0(\delta(gx, gy)), \varphi_0(\delta(gx, Fx)), \varphi_0(\delta(gy, Fy)), \varphi_0(\delta(gx, Fy)), \varphi_0(\delta(Fx, gy))\}. \tag{2}$$

A mapping $\psi_0 : [0, +\infty) \to [0, +\infty)$ defined by

$$\psi(t) = \frac{t + \varphi_0(t)}{2}$$

is upper semicontinuous. Additionally, we have $s\psi \in \Phi_1$ because $s\varphi_0 \in \Phi_1$. By Lemma 4, we obtained that there exists continuous function φ such that $s\varphi \in \Phi_1$ and

$$\delta(Fx, Fy) \leq \max\{\varphi(\delta(gx, gy)), \varphi(\delta(gx, Fx)), \varphi(\delta(gy, Fy)), \varphi(\delta(gx, Fy)), \varphi(\delta(Fx, gy))\}. \tag{3}$$

for every $x, y \in X$. It follows that for any $x, y \in X$ we have

$$\delta(Fx, Fy) \leq \varphi(\max\{\delta(gx, gy), \delta(gx, Fx), \delta(gy, Fy), \delta(gx, Fy), \delta(Fx, gy)\}),$$

because φ is monotone nondecreasing. Thus, we can assume that $\varphi_j = \varphi$ for $j = 1, 2, 3, 4, 5$ and $s\varphi \in \Phi_1$. Let $x_0 \in X$ be arbitrary and let (x_n) be an arbitrary sequence such that $gx_{i+1} \in Fx_i$, for every $i = 0, 1, 2, \ldots$ and (y_n) arbitrary corresponding Jungck sequence with initial point x_0. Let $d_0 = \delta(gx_0, Fx_0)$. We will show that it exists $r_0 \in (0, +\infty)$ such that

$$r_0 - s\varphi(r_0) \leq d_0 < r - s\varphi(r), \tag{4}$$

for $r > r_0$. Let $D = \{r \mid \text{ for all } t > r,\ t - s\varphi(t) > d_0\}$. Since

$$r - \varphi(r) \to +\infty \text{ as } r \to +\infty$$

the set D is nonempty. If $q_1 \in D$ and $q_2 > q_1$ we have $q_2 \in D$. Therefore, D is an unbounded set. Let

$$r_0 = \inf D.$$

Let $n \in \mathbb{N}$ then there exists $r_n \notin D$ such that $r_0 - 1/n < r_n$. So,

$$r_0 > r_n > r_0 - 1/n \text{ such that } r_n - s\varphi(t_n) \leq d_0.$$

Since φ is nondecreasing, we have

$$s\varphi(r_n) \leq s\varphi(r_0).$$

So,

$$r_n - s\varphi(r_0) \leq d_0.$$

Taking a limit as $n \to +\infty$, we obtain

$$r_0 - s\varphi(r_0) \leq d_0.$$

For any $j \geq 0$, define

$$\mathcal{O}_n(x_j) = \{y \in Fx_k \mid k = j, j+1, j+2, \ldots, j+n\}$$

and

$$\mathcal{O}(x_j) = \{y \in Fx_k \mid k = j, j+1, j+2, \ldots\}.$$

Next, we prove that for all positive integers k, n there holds

$$\mathrm{diam}(\mathcal{O}_n(x_k)) \leq \varphi(\mathrm{diam}(\mathcal{O}_{n+1}(x_{k-1}))). \tag{5}$$

Since φ is monotone nondecreasing, it commutes with max for $i, j \in \{k, \ldots, k+n\}$, we have

$$\begin{aligned} \delta(Fx_i, Fx_j) &\leq \varphi(\max\{\delta(gx_i, gx_j), \delta(gx_i, Fx_i), \delta(gx_j, Fx_j), \delta(gx_i, Fx_j), \delta(gx_j, Fx_i)\}) \\ &= \varphi(\max\{\delta(Fx_{i-1}, Fx_{j-1}), \delta(Fx_{i-1}, Fx_i), \delta(Fx_{j-1}, Fx_j), \delta(Fx_{i-1}, Fx_j), \delta(Fx_{j-1}, Fx_i)\}) \\ &\leq \varphi(\mathrm{diam}(\mathcal{O}_{n+1}(x_{k-1}))). \end{aligned}$$

By induction, from (5) we obtain

$$\delta(\mathcal{O}_n(x_k)) \leq \varphi^l(\mathrm{diam}(\mathcal{O}_{n+l}(x_{k-l}))). \tag{6}$$

For $i, j \in \{1, \ldots, n\}$ we have $fx_i, fx_j \in \mathcal{O}_{n-1}(x_1)$. Therefore, by (5) we have

$$d(Fx_i, Fx_j) \leq \mathrm{diam}(\mathcal{O}_{n-1}(x_1)) \leq \varphi(\mathrm{diam}(\mathcal{O}_n(x_0))) < \mathrm{diam}(\mathcal{O}_n(x_0)).$$

Therefore, there exists $k \in \{1, \ldots, n\}$, such that

$$\begin{aligned} \mathrm{diam}(\mathcal{O}_n(x_0)) &= \delta(Fx_0, Fx_k) \leq s[\delta(Fx_0, Fx_1) + \delta(Fx_1, Fx_k)] \leq sd_0 + s\mathrm{diam}(\mathcal{O}_{n-1}(x_1)) \\ &\leq sd_0 + s\varphi(\mathrm{diam}(\mathcal{O}_n(x_0)). \end{aligned}$$

Hence we obtain

$$\mathrm{diam}(\mathcal{O}_n(x_0)) - s\varphi(\mathrm{diam}(\mathcal{O}_n(x_0))) \leq d_0$$

which implies $\mathrm{diam}(\mathcal{O}_n(x_0)) \leq r_0$. So,

$$\mathrm{diam}(\mathcal{O}(x_0)) = \sup_n \mathrm{diam}(\mathcal{O}_n(x_0)) \leq r_0. \tag{7}$$

So, we conclude that each Jungck sequence defined by F and g is bounded. We will show that the corresponding Jungck sequence (y_n) is a Cauchy. Let $m > n$, then $Fx_n, Fx_m \subseteq \mathcal{O}_{m-n+1}(x_n)$, using (6) (with $l = n$) and (7) we obtain

$$d(y_n, y_m) \leq \delta(Fx_n, Fx_m) \leq \mathrm{diam}(\mathcal{O}_{m-n+1}(x_n)) \leq \varphi^n(\mathrm{diam}(\mathcal{O}_{m+1}(x_0))) \leq \varphi^n(r_0) \to 0,$$

as $m, n \to \infty$. Since $F(X) \subseteq g(X)$ and $g(X)$ is complete, we obtain that (y_n) is convergent. Let $y \in X$, such that $\lim_{n\to+\infty} y_n = y$. Since $y \in G(X)$, we conclude that there exists $z \in X$ such that $y = gz$. Let us prove that $y \in F(z)$. Suppose that $\delta(y, Fz) > 0$. By (1) we have

$$\begin{aligned} \delta(y, Fz) &\leq \delta(y, y_{n+1}) + \delta(y_{n+1}, Fz) \\ &\leq \delta(y, y_{n+1}) + \delta(Fx_n, Fz) \\ &\leq \delta(y, y_{n+1}) + \varphi(\max\{\delta(y_n, y), \delta(y_n, Fx_n), \delta(y, Fz), \delta(y_n, Fz), \delta(y, Fx_n)\}). \end{aligned} \tag{8}$$

By (8) we obtain
$$\overline{\lim}\delta(y_n, Fx_n) \leq \overline{\lim}\delta(Fx_{n-1}, Fx_n) = 0 \tag{9}$$
and
$$\overline{\lim}\delta(y, Fx_n) \leq \overline{\lim}[\delta(y, y_n) + \delta(y_n, Fx_n)] \leq \overline{\lim}\delta(y, y_n) + \overline{\lim}\delta(Fx_{n-1}, Fx_n) = 0.$$

Furthermore, we have
$$\overline{\lim}\delta(y_n, Fz) \leq \overline{\lim}[\delta(y_n, y) + \delta(y, Fz)] \leq \overline{\lim}\delta(y_n, y) + \overline{\lim}\delta(y, Fz) = \delta(y, Fz). \tag{10}$$

From (8), (9) and (10) it follows
$$\begin{aligned} \delta(y, Fz) &\leq \overline{\lim}\delta(y, y_{n+1}) + \overline{\lim}\varphi(\max\{\delta(y_n, y), \delta(y_n, Fx_n), \delta(y, Fz), \delta(y_n, Fz), \delta(y, Fx_n)\}) \\ &\leq \varphi(\max\{0, 0, \delta(y, Fz), \delta(y, Fz), 0\}). \end{aligned} \tag{11}$$

Thus, we have
$$d(y, Fz) \leq \varphi(d(y, Fz)),$$
which is a contradiction. Hence $\delta(y, Fz) = 0$. Therefore, $\{y\} = Fz$.

Let is show uniqueness. Suppose
$$Fz = \{g(z)\} = \{y\} \text{ and } F\{z'\} = \{gz'\} = \{y'\}.$$

Then by (1) we have
$$\begin{aligned} d(y, y') &= \delta(Fz, Fz') \\ &\leq \varphi(\max\{d(gz, gz'), d(gz, Fz), d(gz', Fz'), d(gz, Fz'), d(gz', Fz)\}) \\ &= \varphi(\max\{d(y, y'), 0, 0, d(y, y'), d(y', y)\}) = \varphi(d(y, y')) < d(y, y'). \end{aligned} \tag{12}$$

So, $d(y, y') = 0$, the Jungck sequence converges uniquely to the point of coincidence. If $X = Y$ and F and g are weakly compatible, using Lemma 1, we obtain that $y = z$ unique common fixed point of F and g. □

Example 1. *Let $X = [0, +\infty), Y = [0, +\infty), s = 2$ and the mappings*
$$d : Y \times Y \to [0, +\infty), F : X \to \mathcal{B}(Y), g : X \to X$$
defined by
$$d(x, y) = (x - y)^2, F(x) = \left[0, \frac{x}{2}\right], g(x) = 2x.$$
Suppose that the functions $\varphi_i : [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$ defined by
$$\varphi_i(t) = \frac{t}{i+2}, i = 1, 2, 3, 4, 5.$$
Then we have:

(i) *$F(X) = [0, +\infty)$ and $g(X) = [0, +\infty)$, therefore $F(X) \subseteq g(X)$;*
(ii) *$g(X)$ is a complete subspace of Y;*
(iii) *$\varphi_i : [0, +\infty) \to [0, +\infty)$ and $s\varphi_i(t) = \frac{2t}{i+2}$, so $s\varphi_i \in \Phi_2, i = 1, 2, 3, 4, 5$;*
(iv) *$\delta(Fx, Fy) = \delta([0, \frac{x}{2}], [0, \frac{y}{2}]) = \frac{|x-y|}{2}$, $\varphi_1(\delta(gx, gy) = \varphi_1(2|x - y|) = \frac{2|x-y|}{3}$, therefore, condition (1) is satisfied;*
(v) *$X = Y$;*
(vi) *$F(gx) = F(2x) = [0, x]$, $g(Fx) = g([0, \frac{x}{2}]) = [0, x]$, so F and g are weakly compatible.*

Therefore, all the assumptions of Theorem 1 are satisfied and we conclude that F and g have a unique coincidence point.

Example 2. *Let $X = Y = [0, +\infty)$, $s = 2$ and the mappings*

$$d : X \times X \to [0, +\infty), F : X \to \mathcal{B}(Y), g : X \to X$$

defined by $d(x, y) = (x - y)^2$, $Fx = \{\frac{x}{3^k} : k \in \mathbb{N}\}$, $gx = 3x$. Let the functions $\varphi_i : [0, +\infty) \to [0, +\infty)$ defined by $\varphi_i(t) = \frac{t}{2}$, $i = 1, 2, 3, 4, 5$. Then we have:

(i) $F(X) \subseteq [0, +\infty), x \in [0, +\infty)$ *and* $g(X) = [0, +\infty)$, *therefore,* $F(X) \subseteq g(X)$;

(ii) $g(X)$ *is a complete subspace of* Y;

(iii) $\varphi_i : [0, +\infty) \to [0, +\infty)$ *and* $s\varphi_i \in \Phi_2$;

(iv)

$$\begin{aligned}
\delta(Fx, Fy) &= \delta(\{\frac{x}{3^k} : k \in \mathbb{N}\}, \{\frac{y}{3^k} : k \in \mathbb{N}\}) = \max\{\frac{x^2}{9}, \frac{y^2}{9}\} \\
&\leq \max\{\frac{9x^2}{2}, \frac{9y^2}{2}\} = \max\{\varphi_4(\delta(gx, Fy)), \varphi_5(\delta(gy, Fx))\} \\
&\leq \max\{\varphi_1(\delta(gx, gy)), \varphi_2(\delta(gx, Fx)), \varphi_3(\delta(gy, Fy)), \varphi_4(\delta(gx, Fy)), \varphi_5(\delta(Fx, gy)\}
\end{aligned}$$

therefore, condition (1) is satisfied;

(v) $X = Y$;

(vi) $F(gx) = F(3x) = \{\frac{x}{3^{k-1}} : k \in \mathbb{N}\} = g(Fx)$, *therefore F and g are weakly compatible.*

So, F and g have a unique coincidence point.

Remark 2. *Theorem 1 extended earlier results for nonlinear contractions on metric space obtained by J. Daneš [4], A. A. Ivanov [5], I. Aranđelović, M. Rajović and V. Kilibarda [6] and M. Bessenyei [7], common fixed-point results of K. M. Das, K. V. Naik [9] and C. Di Bari and P. Vetro [10]. Please note that contractive condition (1) was defined earlier by several authors; see for example [14].*

4. The Common Endpoints for Hybrid Dynamical System

For an ordered pair (X, T) we say that it is a set-valued dynamical system, where X is given space and $T : X \to 2^X$ a multivalued map.

If T is a single-valued mapping, we obtain the usual dynamical system.

If x is a strict fixed point of T, we say that x is an endpoint of dynamical system.

A sequence (x_n) in X defined by $x_n \in T(x_{n-1})$ is called a dynamical process or trajectory; see more about dynamical systems in the famous monographs [40,41].

Definition 4. *Let X be a nonempty set, $T : X \to 2^X$ and $g : X \to X$. Then (X, T, g) is a hybrid dynamical system.*

For a unique point of coincidence for T and g, we say that the endpoint (or stationary point) for hybrid dynamical system (X, T, g).

A sequence (x_n) in X defined by $x_n \in T(x_{n-1})$ is called a dynamical process or trajectory of the hybrid system (X, T, g).

From Theorem 1, we obtained the following result.

Theorem 2. *Let (X, F, g) be hybrid dynamical system, where (X, d, s) is a complete b-metric space and $F : X \to \mathcal{B}(X)$. Suppose that $F(X) \subseteq g(X)$ and that $g(X)$ is a complete subspace of Y. If there exists $\varphi_i \in \Phi_2$, $i = 1, 2, 3, 4, 5$ such that*

$$\delta(Fx, Fy) \leq \max\{\varphi_1(d(gx, gy)), \varphi_2(\delta(gx, Fx)), \varphi_3(\delta(gy, Fy)), \varphi_4(\delta(gx, Fy)), \varphi_5(\delta(F(x, gy))\}, \tag{13}$$

for any $x, y \in X$, then hybrid dynamic system (X, F, g) has an endpoint which is the limit of every dynamical process defined by F and g.

Therefore, from Theorem 1 we obtain the result for a dynamical system.

Let X be a Hausdorff topological linear space. If there exists (see Köte [42]) a continuous function $\|.| : \mathrm{X} \to [0, +\infty)$ such that:

(1) $\|\mathrm{x}| \geq 0$;
(2) $\|\mathrm{x}| = 0$ if and only if $\mathrm{x} = 0$;
(3) $\|t\mathrm{x}| = |t|\|\mathrm{x}|$;
(4) there exists $s \geq 1$ such that $\|\mathrm{x} + \mathrm{y}| \leq s(\|\mathrm{x}| + \|\mathrm{y}|)$,

for all $\mathbf{x}, \mathrm{y} \in \mathrm{X}, t \in \mathbb{R}$, then $(\mathrm{X}, \|.|, s)$ is the quasi-normed space. Mapping $\|.|$ is said to be a quasi-norm.

If $(\mathrm{X}, \|.|, s)$ is the quasi-normed space then (X, d, s) is b-metric space, where $d(\mathrm{x}, \mathrm{y}) = \|.|$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$. If (X, d, s) is complete then $(\mathrm{X}, \|.|, s)$ is a quasi-Banach space.

Let $I \subseteq \mathbb{R}$ be an interval, and $\mathbb{S}$ be a set of functions defined on I, $X \subseteq \mathbb{S}$, $\|.| : X \to [0, +\infty)$ and $s \geq 1$ such that $(X, \|.|, s)$ is quasi-Banach space, and $f : X \times I \to X$. Then equation (see [43]),

$$\dot{x}(t) = f(x(t), u(t)), \tag{14}$$

where $u : I \to I$ is the known control function, $x : X \to X$ is the unknown Gateaux differentiable function (so-called state function) and the $\dot{x}$ Gateaux derivate of x is equivalent with differential inclusion

$$\dot{x}(t) \in F(x(t)),$$

for some $F : X \to 2^X$. Therefore, for $g(x) = \dot{x}$ from Theorem 2 we obtained sufficient conditions for the existence of an endpoint of a hybrid dynamical system (X, F, g), which is also a solution of (14).

5. Conclusions

We obtained new results for the points of coincidence and fixed points in the hybrid pair of multivalued and single-valued mappings in b-metric spaces. We introduced five new nonlinear comparison functions. Our results generalize and improve several recent results in the literature. We also present the application of the obtained results in dynamical systems. We believe that our main result can be a starting point for new research in other generalized metric spaces.

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