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On Some New Jungck–Fisher–Wardowski Type Fixed Point Results

Jelena Vujaković [1,*], Eugen Ljajko [1], Slobodan Radojević [2] and Stojan Radenović [2]

1 Department of Mathematics, Faculty of Sciences, University of Priština-Kosovska Mitrovica, Lole Ribara 29, 38200 Kosovska Mitrovica, Serbia; eugen.ljajko@pr.ac.rs
2 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia; sradojevic@mas.bg.ac.rs (S.R.); radens@beotel.net (S.R.)
* Correspondence: jelena.vujakovic@pr.ac.rs

Received: 11 November 2020; Accepted: 8 December 2020; Published: 10 December 2020

Abstract: Many authors used the concept of $\mathcal{F}-$contraction introduced by Wardowski in 2012 in order to define and prove new results on fixed points in complete metric spaces. In some later papers (for example Proinov P.D., J. Fixed Point Theory Appl. (2020)22:21, doi:10.1007/s11784-020-0756-1) it is shown that conditions (F2) and (F3) are not necessary to prove Wardowski's results. In this article we use a new approach in proving that the Picard–Jungck sequence is a Cauchy one. It helps us obtain new Jungck–Fisher–Wardowski type results using Wardowski's condition (F1) only, but in a way that differs from the previous approaches. Along with that, we came to several new contractive conditions not known in the fixed point theory so far. With the new results presented in the article, we generalize, extend, unify and enrich methods presented in the literature that we cite.

Keywords: banach contraction principle; Fisher fixed point theorem; Wardowski-type contractions; compatible; weakly compatible; common fixed point

MSC: 47H10; 54H25

1. Introduction and Preliminaries

In 1976, Jungck [1] proved the following result.

Theorem 1. *Let $\mathcal{T}$ and $\mathcal{I}$ be commuting mappings of a complete metric space (Υ, d_Υ) into itself that satisfy the inequality*

$$d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{T}\widetilde{y}) \leq \lambda d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{I}\widetilde{y}) \tag{1}$$

for all $\widetilde{x}, \widetilde{y} \in \Upsilon$, where $0 < \lambda < 1$. If the range of $\mathcal{I}$ contains the range of $\mathcal{T}$ and if $\mathcal{I}$ is continuous, then $\mathcal{T}$ and $\mathcal{I}$ have a unique common fixed point.

In 1981, Fisher [2] proved the common fixed point theorem for four mappings and thus obtained a genuine generalization of Jungck's result from 1976.

Theorem 2. *Let $\mathcal{S}, \mathcal{I}$ and $\mathcal{T}, \mathcal{J}$ be pairs of commuting mappings of a complete metric space (Υ, d_Υ) into itself that satisfies*

$$d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) \leq \lambda d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}) \tag{2}$$

for all $\widetilde{x}, \widetilde{y} \in \Upsilon$, where $0 < \lambda < 1$. If $\mathcal{S}\widetilde{x} \in \mathcal{J}(\Upsilon)$ and $\mathcal{T}\widetilde{x} \in \mathcal{I}(\Upsilon)$ for each $\widetilde{x} \in \Upsilon$ and if $\mathcal{I}$ and $\mathcal{J}$ are continuous, then all mappings $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$ have a unique common fixed point.

Remark 1. *It is obvious that both previous Theorems holds true for $\lambda = 0$ alike. In addition, it is evident that Theorems 1 and 2 genuinely generalize the famous Banach contraction principle* [3].

Since 2012., several research papers (for example [4–16]) considered a new type of contraction mapping introduced by Wardowski [17] . For other new-old types of contractive mappings see e.g., [18–22]. Firstly, in [17] the author introduced the following:

Definition 1. *Let $\mathcal{F}\colon (0,+\infty) \to (-\infty,+\infty)$ be a mapping satisfying:*

$(F1)$ *$\mathcal{F}$ is strictly increasing, i.e., for all $\mathfrak{a},\mathfrak{b} \in (0,+\infty)$, if $\mathfrak{a} < \mathfrak{b}$ then $\mathcal{F}(\mathfrak{a}) < \mathcal{F}(\mathfrak{b})$;*
$(F2)$ *For each sequence $\{\widetilde{x}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathbb{N}} \subset (0,+\infty)$, $\lim_{\mathfrak{p}\to+\infty} \widetilde{x}_{\mathfrak{p}} = 0$ if and only if $\lim_{\mathfrak{p}\to+\infty} \mathcal{F}(\widetilde{x}_{\mathfrak{p}}) = -\infty$;*
$(F3)$ *There exists $\mathfrak{m} \in (0,1)$ such that $\lim_{\widetilde{x}\to 0^+} \widetilde{x}^{\mathfrak{m}} \mathcal{F}(\widetilde{x}) = 0$.*

A self-mapping $\mathcal{A}$ of a complete metric space $(\mathrm{Y}, d_{\mathrm{Y}})$ into itself is said to be an $\mathcal{F}-$contraction if there exists $\tau > 0$ such that

$$d_{\mathrm{Y}}(\mathcal{A}\widetilde{x},\mathcal{A}\widetilde{y}) > 0 \text{ implies } \tau + \mathcal{F}(d_{\mathrm{Y}}(\mathcal{A}\widetilde{x},\mathcal{A}\widetilde{y})) \le \mathcal{F}(d_{\mathrm{Y}}(\widetilde{x},\widetilde{y})), \tag{3}$$

for all $\widetilde{x},\widetilde{y} \in \mathrm{Y}$

Remark 2. *Since inequality $\mathcal{F}(t-0) \le \mathcal{F}(t) \le \mathcal{F}(t+0)$ holds for all $t \in (0,+\infty)$, one can conclude (using* (F1) *property only) that there are $\lim_{c\to t^-} \mathcal{F}(c) = \mathcal{F}(t-0)$ and $\lim_{c\to t^+} \mathcal{F}(c) = \mathcal{F}(t+0)$.*

In addition, from property (F1) it follows either

(1) $\mathcal{F}(0+0) = \lim_{\widetilde{x}\to 0^+} \mathcal{F}(\widetilde{x}) = \mathfrak{m}, \mathfrak{m} \in (-\infty,+\infty)$, or
(2) $\mathcal{F}(0+0) = \lim_{\widetilde{x}\to 0^+} \mathcal{F}(\widetilde{x}) = -\infty$ (for more details see [13,23]).

Additionally, in [17] Wardowski proved and generalized the Banach contraction principle in the following form:

Theorem 3. *Let $(\mathrm{Y}, d_{\mathrm{Y}})$ be a complete metric space and $\mathcal{A}\colon \mathrm{Y} \to \mathrm{Y}$ an $\mathcal{F}-$ contraction. Then $\mathcal{A}$ has a unique fixed point, say $\widetilde{x}^*$ in Y and for every $\widetilde{x} \in \mathrm{Y}$ the sequence $\{\mathcal{A}^{\mathfrak{p}}\widetilde{x}\}$, $\mathfrak{p} \in \mathbb{N}$ converges to $\widetilde{x}^*$.*

Note that in 2013., Turinci [24] noticed that condition (F2) can be weakened as follows:

(T) $\lim_{t\to 0^+} \mathcal{F}(t) = -\infty$.

Other details of property (F2) can be found in Secelean's work ([12] [Lemma 2 and Remark 3.1]). Further, Wardowski in [15] introduced a concept of $(\tau,\mathcal{F})-$contraction on metric space. A self mapping $\mathcal{A}\colon \mathrm{Y} \to \mathrm{Y}$ is said to be $(\tau,\mathcal{F})-$contraction if for some $\mathcal{F}\colon (0,+\infty) \to (-\infty,+\infty)$ and $\tau\colon (0,+\infty) \to (0,+\infty)$ the following conditions apply

$(\tau 1)$ $\mathcal{F}$ satisfies (F1) and (T);
$(\tau 2)$ $\liminf_{c\to t^+} \tau(c) > 0$ for all $t \ge 0$;
$(\tau 3)$ $\tau(d_{\mathrm{Y}}(\widetilde{x},\widetilde{y})) + \mathcal{F}(d_{\mathrm{Y}}(\mathcal{A}\widetilde{x},\mathcal{A}\widetilde{y})) \le \mathcal{F}(d_{\mathrm{Y}}(\widetilde{x},\widetilde{y}))$ for all $\widetilde{x},\widetilde{y} \in \mathrm{Y}$ such that $\mathcal{A}\widetilde{x} \ne \mathcal{A}\widetilde{y}$.

Among other things, he generalized the result from [17] and proved the following theorem [15].

Theorem 4. *Let $\mathcal{A}\colon \mathrm{Y} \to \mathrm{Y}$ be a $(\tau,\mathcal{F})-$contraction on complete metric space $(\mathrm{Y}, d_{\mathrm{Y}})$. Then $\mathcal{A}$ has a unique fixed point.*

Recently, in [14], we proved the Theorem 4 using only the condition (F1) and the following two lemmas [9,10] .

Lemma 1. *Let* $\{\widetilde{x}_{\mathfrak{p}}\}$ *be a sequence in a metric space* $(\mathrm{Y}, d_{\mathrm{Y}})$ *such that* $\lim_{\mathfrak{p}\to+\infty} d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}}, \widetilde{x}_{\mathfrak{p}+1}\right) = 0$. *If* $\{\widetilde{x}_{\mathfrak{p}}\}$ *is not a Cauchy sequence, then there exist* $\varepsilon > 0$ *and two sequences* $\{\mathfrak{p}(\mathfrak{s})\}$ *and* $\{\mathfrak{q}(\mathfrak{s})\}$ *of positive integers such that* $\mathfrak{p}(\mathfrak{s}) > \mathfrak{q}(\mathfrak{s}) > \mathfrak{s}$ *and the sequences:*

$$\left\{d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}(\mathfrak{s})}, \widetilde{x}_{\mathfrak{q}(\mathfrak{s})}\right)\right\}, \left\{d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}(\mathfrak{s})+1}, \widetilde{x}_{\mathfrak{q}(\mathfrak{s})}\right)\right\}, \left\{d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}(\mathfrak{s})}, \widetilde{x}_{\mathfrak{q}(\mathfrak{s})-1}\right)\right\},$$
$$\left\{d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}(\mathfrak{s})+1}, \widetilde{x}_{\mathfrak{q}(\mathfrak{s})-1}\right)\right\}, \left\{d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}(\mathfrak{s})+1}, \widetilde{x}_{\mathfrak{q}(\mathfrak{s})+1}\right)\right\}, \tag{4}$$

tend to ε^+, *as* $\mathfrak{s} \to +\infty$.

Lemma 2. *Let* $\{\widetilde{x}_{\mathfrak{p}+1}\} = \{\mathcal{A}\widetilde{x}_{\mathfrak{p}}\} = \{\mathcal{A}^{\mathfrak{p}}\widetilde{x}_0\}, \mathfrak{p} \in \mathbb{N} \cup \{0\}, \mathcal{A}^0\widetilde{x}_0 = \widetilde{x}_0$ *be a Picard sequence in a metric space* $(\mathrm{Y}, d_{\mathrm{Y}})$ *induced by a mapping* $\mathcal{A}\colon \mathrm{Y} \to \mathrm{Y}$ *and let* $\widetilde{x}_0 \in \mathrm{Y}$ *be an initial point. If* $d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}}, \widetilde{x}_{\mathfrak{p}+1}\right) < d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}-1}, \widetilde{x}_{\mathfrak{p}}\right)$ *for all* $\mathfrak{p} \in \mathbb{N}$ *then* $\widetilde{x}_{\mathfrak{p}} \neq \widetilde{x}_{\mathfrak{q}}$ *whenever* $\mathfrak{p} \neq \mathfrak{q}$.

Proof. Suppose the opposite, let $\widetilde{x}_{\mathfrak{p}} = \widetilde{x}_{\mathfrak{q}}$ for some $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}$ with $\mathfrak{p} < \mathfrak{q}$. Then $\widetilde{x}_{\mathfrak{p}+1} = \mathcal{A}\widetilde{x}_{\mathfrak{p}} = \mathcal{A}\widetilde{x}_{\mathfrak{q}} = \widetilde{x}_{\mathfrak{q}+1}$. Further, we get

$$d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}}, \widetilde{x}_{\mathfrak{p}+1}\right) = d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{q}}, \widetilde{x}_{\mathfrak{q}+1}\right) < d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{q}-1}, \widetilde{x}_{\mathfrak{q}}\right) < \ldots < d_{\mathrm{Y}}\left(\widetilde{x}_{\mathfrak{p}}, \widetilde{x}_{\mathfrak{p}+1}\right), \tag{5}$$

and that is a contradiction. □

At the end of this section, let us recall the following terms and results (for more information, see [25,26]). Let $\mathcal{A}$ and $\mathcal{B}$ be self mappings of a nonempty set Y. If $\widetilde{y} = \mathcal{A}\widetilde{x} = \mathcal{B}\widetilde{x}$ for some $\widetilde{x} \in \mathrm{Y}$, then $\widetilde{x}$ is called a coincidence point of $\mathcal{A}$ and $\mathcal{B}$, and $\widetilde{y}$ is called a point of coincidence of $\mathcal{A}$ and $\mathcal{B}$. A pair of self mappings $(\mathcal{A}, \mathcal{B})$ is said to be a compatible if $\lim_{\mathfrak{p}\to+\infty} d_{\mathrm{Y}}\left(\mathcal{A}\mathcal{B}\widetilde{x}_{\mathfrak{p}}, \mathcal{B}\mathcal{A}\widetilde{x}_{\mathfrak{p}}\right) = 0$, for every sequence $\{\widetilde{x}_{\mathfrak{p}}\}$ in Y for which $\lim_{\mathfrak{p}\to+\infty} \mathcal{A}\widetilde{x}_{\mathfrak{p}} = \lim_{\mathfrak{p}\to+\infty} \mathcal{B}\widetilde{x}_{\mathfrak{p}} = t$, for some $t \in \mathrm{Y}$. The pair $(\mathcal{A}, \mathcal{B})$ is weakly compatible if mappings $\mathcal{A}$ and $\mathcal{B}$ commute at their coincidence points. A sequence $\{\widetilde{x}_{\mathfrak{p}}\}$ in Y is said to be a Picard–Jungck sequence of the pair $(\mathcal{A}, \mathcal{B})$ (based on $\widetilde{x}_0$) if $\widetilde{y}_{\mathfrak{p}} = \mathcal{A}\widetilde{x}_{\mathfrak{p}} = \mathcal{B}\widetilde{x}_{\mathfrak{p}+1}$ for all $\mathfrak{p} \in \mathbb{N} \cup \{0\}$.

Proposition 1. [26] *If weakly compatible self mappings* $\mathcal{A}$ *and* $\mathcal{B}$ *of a set* Y *have a unique point of coincidence* $\widetilde{y} = \mathcal{A}\widetilde{x} = \mathcal{B}\widetilde{x}$, *then* $\widetilde{y}$ *is a unique common fixed point of* $\mathcal{A}$ *and* $\mathcal{B}$.

2. Results

In the following theorem, we bring forward first of our results for four self-mappings in a complete metric space.

Theorem 5. *Let* $(\mathcal{S}, \mathcal{I})$ *and* $(\mathcal{T}, \mathcal{J})$ *be a pair of compatible self-mappings of a complete metric space* $(\mathrm{Y}, d_{\mathrm{Y}})$ *into itself and* $\mathcal{F}\colon (0, +\infty) \to (-\infty, +\infty)$ *is a strictly increasing mapping such that*

$$\tau + \mathcal{F}\left(d_{\mathrm{Y}}\left(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}\right)\right) \leq \mathcal{F}\left(\mathfrak{M}^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{T}}\left(\widetilde{x}, \widetilde{y}\right)\right), \tag{6}$$

for all $\widetilde{x}, \widetilde{y} \in \mathrm{Y}$ *with* $d_{\mathrm{Y}}\left(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}\right) > 0$, *where*

$$\mathfrak{M}^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{T}}\left(\widetilde{x}, \widetilde{y}\right) = \max\left\{d_{\mathrm{Y}}\left(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}\right), d_{\mathrm{Y}}\left(\mathcal{S}\widetilde{x}, \mathcal{I}\widetilde{x}\right), d_{\mathrm{Y}}\left(\mathcal{T}\widetilde{y}, \mathcal{J}\widetilde{y}\right), \frac{d_{\mathrm{Y}}\left(\mathcal{S}\widetilde{x}, \mathcal{J}\widetilde{y}\right) + d_{\mathrm{Y}}\left(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{x}\right)}{2}\right\},$$

and τ *is a given positive constant. If* $\mathcal{I}, \mathcal{J}, \mathcal{S}$ *and* $\mathcal{T}$ *are continuous and if* $\mathcal{S}(\mathrm{Y}) \subseteq \mathcal{J}(\mathrm{Y}), \mathcal{T}(\mathrm{Y}) \subseteq \mathcal{I}(\mathrm{Y})$ *then mappings* $\mathcal{I}, \mathcal{J}, \mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point.*

Proof. First of all we show the uniqueness of a possible common fixed point. Suppose that $\mathcal{I}, \mathcal{J}, \mathcal{S}$ and $\mathcal{T}$ have two distinct common fixed points $\widetilde{x}$ and $\widetilde{y}$ in Y. Since $d_Y(\widetilde{x}, \widetilde{y}) = d_Y(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) > 0$ we get according to (6):

$$\tau + \mathcal{F}(d_Y(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y})) \leq \mathcal{F}\left(\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}(\widetilde{x}, \widetilde{y})\right), \tag{7}$$

where

$$\begin{aligned}\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}(\tilde{x}, \tilde{y}) &= \max\left\{d_Y(\mathcal{I}\tilde{x}, \mathcal{J}\tilde{y}), d_Y(\mathcal{S}\tilde{x}, \mathcal{I}\tilde{x}), d_Y(\mathcal{T}\tilde{y}, \mathcal{J}\tilde{y}), \frac{d_Y(\mathcal{S}\tilde{x}, \mathcal{J}\tilde{y}) + d_Y(\mathcal{T}\tilde{y}, \mathcal{I}\tilde{x})}{2}\right\} \\ &= \max\left\{d_Y(\tilde{x}, \tilde{y}), d_Y(\tilde{x}, \tilde{x}), d_Y(\tilde{y}, \tilde{y}), \frac{d_Y(\tilde{x}, \tilde{y}) + d_Y(\tilde{y}, \tilde{x})}{2}\right\} = d_Y(\tilde{x}, \tilde{y}).\end{aligned}$$

Hence,

$$\tau + \mathcal{F}(d_Y(\widetilde{x}, \widetilde{y})) \leq \mathcal{F}(d_Y(\widetilde{x}, \widetilde{y})). \tag{8}$$

Since $\tau > 0$ and $\widetilde{x} \neq \widetilde{y}$ we get a contradiction. So, if there exists a common fixed point, it is unique. We further prove the existence of this common fixed point. Let $\widetilde{x}_0 \in Y$ be arbitrary. Since $\mathcal{S}\widetilde{x}_0 \in \mathcal{J}(Y)$, there is $\widetilde{x}_1 \in Y$ such that $\mathcal{J}\widetilde{x}_1 = \mathcal{S}\widetilde{x}_0$, and also as $\mathcal{T}\widetilde{x}_1 \in \mathcal{I}(Y)$, let $\widetilde{x}_2 \in Y$ be such that $\mathcal{I}\widetilde{x}_2 = \mathcal{T}\widetilde{x}_1$. In general, there are $\widetilde{x}_{2\mathfrak{p}+1}$ and $\widetilde{x}_{2\mathfrak{p}+2}$ in Y such that $\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1} = \mathcal{S}\widetilde{x}_{2\mathfrak{p}}$ and $\mathcal{I}\widetilde{x}_{2\mathfrak{p}+2} = \mathcal{T}\widetilde{x}_{2\mathfrak{p}+1}$, $\mathfrak{p} = 0, 1, 2, \ldots$. Denote a sequence $\{\widetilde{z}_\mathfrak{p}\}$ with

$$\begin{aligned}\widetilde{z}_{2\mathfrak{p}} &= \mathcal{J}\widetilde{x}_{2\mathfrak{p}+1} = \mathcal{S}\widetilde{x}_{2\mathfrak{p}} \\ \widetilde{z}_{2\mathfrak{p}+1} &= \mathcal{I}\widetilde{x}_{2\mathfrak{p}+2} = \mathcal{T}\widetilde{x}_{2\mathfrak{p}+1},\end{aligned}$$

$\mathfrak{p} = 0, 1, 2, \ldots$. We will show that $\{\widetilde{z}_\mathfrak{p}\}$ is a Cauchy sequence. Due to the condition $d_Y(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) > 0$ it follows that $d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1}) > 0$ for all $\widetilde{x}, \widetilde{y} \in Y$ and $\mathfrak{p} \in \mathbb{N} \cup \{0\}$. Replacing $\widetilde{x}$ and $\widetilde{y}$ respectively with $\widetilde{x}_{2\mathfrak{p}}$ and $\widetilde{x}_{2\mathfrak{p}+1}$ in (6) we obtain

$$\tau + \mathcal{F}(d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1})) \leq \mathcal{F}\left(\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}(\widetilde{x}_{2\mathfrak{p}}, \widetilde{x}_{2\mathfrak{p}+1})\right), \tag{9}$$

where

$$\begin{aligned}\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}(\tilde{x}_{2\mathfrak{p}}, \tilde{x}_{2\mathfrak{p}+1}) &= \max\Big\{d_Y(\mathcal{I}\tilde{x}_{2\mathfrak{p}}, \mathcal{J}\tilde{x}_{2\mathfrak{p}+1}), d_Y(\mathcal{S}\tilde{x}_{2\mathfrak{p}}, \mathcal{I}\tilde{x}_{2\mathfrak{p}}), d_Y(\mathcal{T}\tilde{x}_{2\mathfrak{p}+1}, \mathcal{J}\tilde{x}_{2\mathfrak{p}+1}), \\ &\qquad \frac{d_Y(\mathcal{S}\tilde{x}_{2\mathfrak{p}}, \mathcal{J}\tilde{x}_{2\mathfrak{p}+1}) + d_Y(\mathcal{T}\tilde{x}_{2\mathfrak{p}+1}, \mathcal{I}\tilde{x}_{2\mathfrak{p}})}{2}\Big\} \\ &= \max\Big\{d_Y(\tilde{z}_{2\mathfrak{p}-1}, \tilde{z}_{2\mathfrak{p}}), d_Y(\tilde{z}_{2\mathfrak{p}}, \tilde{z}_{2\mathfrak{p}-1}), d_Y(\tilde{z}_{2\mathfrak{p}+1}, \tilde{z}_{2\mathfrak{p}}), \\ &\qquad \frac{d_Y(\tilde{z}_{2\mathfrak{p}}, \tilde{z}_{2\mathfrak{p}}) + d_Y(\tilde{z}_{2\mathfrak{p}+1}, \tilde{z}_{2\mathfrak{p}-1})}{2}\Big\} \\ &\leq \max\left\{d_Y(\tilde{z}_{2\mathfrak{p}-1}, \tilde{z}_{2\mathfrak{p}}), d_Y(\tilde{z}_{2\mathfrak{p}+1}, \tilde{z}_{2\mathfrak{p}}), \frac{0 + d_Y(\tilde{z}_{2\mathfrak{p}+1}, \tilde{z}_{2\mathfrak{p}}) + d_Y(\tilde{z}_{2\mathfrak{p}}, \tilde{z}_{2\mathfrak{p}-1})}{2}\right\} \\ &\leq \max\{d_Y(\tilde{z}_{2\mathfrak{p}-1}, \tilde{z}_{2\mathfrak{p}}), d_Y(\tilde{z}_{2\mathfrak{p}}, \tilde{z}_{2\mathfrak{p}+1})\}.\end{aligned}$$

Hence, (9) transforms into

$$\tau + \mathcal{F}(d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1})) \leq \mathcal{F}(\max\{d_Y(\widetilde{z}_{2\mathfrak{p}-1}, \widetilde{z}_{2\mathfrak{p}}), d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1})\}). \tag{10}$$

It is clear that $\max\{d_Y(\widetilde{z}_{2\mathfrak{p}-1}, \widetilde{z}_{2\mathfrak{p}}), d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1})\} = d_Y(\widetilde{z}_{2\mathfrak{p}-1}, \widetilde{z}_{2\mathfrak{p}})$. Finally, since $\mathcal{F}$ is a strictly increasing mapping and $d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1}) < d_Y(\widetilde{z}_{2\mathfrak{p}-1}, \widetilde{z}_{2\mathfrak{p}})$ for all $\mathfrak{p} \in \mathbb{N}$ we have

$$\tau + \mathcal{F}(d_Y(\widetilde{z}_{2\mathfrak{p}}, \widetilde{z}_{2\mathfrak{p}+1})) \leq \mathcal{F}(d_Y(\widetilde{z}_{2\mathfrak{p}-1}, \widetilde{z}_{2\mathfrak{p}})). \tag{11}$$

Similarly, replacing $\widetilde{x}$ with $\widetilde{x}_{2\mathfrak{p}+2}$ and $\widetilde{y}$ with $\widetilde{x}_{2\mathfrak{p}+1}$ in (6), it follows $d_{\mathrm{Y}}\left(\widetilde{z}_{2\mathfrak{p}+2},\widetilde{z}_{2\mathfrak{p}+1}\right) < d_{\mathrm{Y}}\left(\widetilde{z}_{2\mathfrak{p}+1},\widetilde{z}_{2\mathfrak{p}}\right)$, for all $\mathfrak{p}\in\mathbb{N}$. So,

$$d_{\mathrm{Y}}\left(\widetilde{z}_{\mathfrak{p}+1},\widetilde{z}_{\mathfrak{p}}\right) < d_{\mathrm{Y}}\left(\widetilde{z}_{\mathfrak{p}},\widetilde{z}_{\mathfrak{p}-1}\right) \tag{12}$$

for all $\mathfrak{p}\in\mathbb{N}$, which, further, implies that $\lim_{\mathfrak{p}\to+\infty} d_{\mathrm{Y}}\left(\widetilde{z}_{\mathfrak{p}+1},\widetilde{z}_{\mathfrak{p}}\right) = d^*_{\mathrm{Y}} \geq 0$. If $d^*_{\mathrm{Y}} > 0$ from (11) follows

$$\tau + \mathcal{F}\left(d^*_{\mathrm{Y}}+0\right) \leq \mathcal{F}\left(d^*_{\mathrm{Y}}+0\right),$$

and that is a contradiction. Hence, $\lim_{\mathfrak{p}\to+\infty} d_{\mathrm{Y}}\left(\widetilde{z}_{\mathfrak{p}+1},\widetilde{z}_{\mathfrak{p}}\right) = 0$. To prove that $\{\widetilde{z}_{\mathfrak{p}}\}$ is a Cauchy sequence, it suffices proving that for the sequence $\{\widetilde{z}_{2\mathfrak{p}}\}$. Indeed, according to Lemma 1, puting $\widetilde{x} = \widetilde{x}_{2\mathfrak{p}(\mathfrak{s})}, \widetilde{y} = \widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}$ in (6), we get

$$\tau + \mathcal{F}\left(d_{\mathrm{Y}}\left(\mathcal{S}\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})},\mathcal{T}\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}\right)\right) \leq \mathcal{F}\left(\mathfrak{M}^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{T}}\left(\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})},\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}\right)\right), \tag{13}$$

where

$$\begin{aligned}\mathfrak{M}^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{T}}(\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})},\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}) &= \max\Big\{d_{\mathrm{Y}}(\mathcal{I}\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})},\mathcal{J}\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}), d_{\mathrm{Y}}(\mathcal{S}\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})},\mathcal{I}\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})}, d_{\mathrm{Y}}(\mathcal{T}\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1},\mathcal{J}\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}),\\ &\qquad \frac{d_{\mathrm{Y}}(\mathcal{S}\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})},\mathcal{J}\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1}) + d_{\mathrm{Y}}(\mathcal{T}\widetilde{x}_{2\mathfrak{q}(\mathfrak{s})+1},\mathcal{I}\widetilde{x}_{2\mathfrak{p}(\mathfrak{s})})}{2}\Big\}\\ &= \max\Big\{d_{\mathrm{Y}}(\widetilde{z}_{2\mathfrak{p}(\mathfrak{s})-1},\widetilde{z}_{2\mathfrak{q}(\mathfrak{s})}), d_{\mathrm{Y}}(\widetilde{z}_{2\mathfrak{p}(\mathfrak{s})},\widetilde{z}_{2\mathfrak{q}(\mathfrak{s})-1}), d_{\mathrm{Y}}(\widetilde{z}_{2\mathfrak{q}(\mathfrak{s})+1},\widetilde{z}_{2\mathfrak{q}(\mathfrak{s})}),\\ &\qquad \frac{d_{\mathrm{Y}}(\widetilde{z}_{2\mathfrak{p}(\mathfrak{s})},\widetilde{z}_{2\mathfrak{q}(\mathfrak{s})}) + d_{\mathrm{Y}}(\widetilde{z}_{2\mathfrak{q}(\mathfrak{s})+1},\widetilde{z}_{2\mathfrak{p}(\mathfrak{s})-1})}{2}\Big\}\\ &\to \max\left\{\varepsilon,\varepsilon,0,\frac{\varepsilon+\varepsilon}{2}\right\} = \varepsilon,\end{aligned}$$

when $\mathfrak{s}\to+\infty$. Taking the limit in (13) as $\mathfrak{s}\to+\infty$, we get a contradiction

$$\tau + \mathcal{F}\left(\varepsilon+0\right) \leq \mathcal{F}\left(\varepsilon+0\right). \tag{14}$$

Thus, $\{\widetilde{z}_{\mathfrak{p}}\}$ is a Cauchy sequence in a complete metric space $(\mathrm{Y}, d_{\mathrm{Y}})$. Having in mind that Y is a complete, we conclude that there exists $\widetilde{z}\in\mathrm{Y}$ such that $\lim_{\mathfrak{p}\to+\infty}\widetilde{z}_{\mathfrak{p}} = \widetilde{z}$ or

$$\lim_{\mathfrak{p}\to+\infty}\mathcal{I}\widetilde{x}_{2\mathfrak{p}+2} = \lim_{\mathfrak{p}\to+\infty}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1} = \lim_{\mathfrak{p}\to+\infty}\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1} = \lim_{\mathfrak{p}\to+\infty}\mathcal{S}\widetilde{x}_{2\mathfrak{p}} = \widetilde{z}.$$

Further, due to the continuity and compatibility of mappings $\mathcal{J}$ and $\mathcal{T}$, we obtain

$$\begin{aligned}d_{\mathrm{Y}}\left(\mathcal{J}\widetilde{z},\mathcal{T}\widetilde{z}\right) &\leq d_{\mathrm{Y}}\left(\mathcal{J}\widetilde{z},\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1}\right) + d_{\mathrm{Y}}\left(\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1},\mathcal{T}\widetilde{z}\right)\\ &\leq d_{\mathrm{Y}}\left(\mathcal{J}\widetilde{z},\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1}\right) + d_{\mathrm{Y}}\left(\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1},\mathcal{T}\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1}\right) + d_{\mathrm{Y}}\left(\mathcal{T}\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1},\mathcal{T}\widetilde{z}\right)\\ &\to 0+0+0 = 0\end{aligned}$$

as $\mathfrak{p}\to+\infty$, because $\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1}\to\widetilde{z}$ implies $\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1}\to\mathcal{J}\widetilde{z}$ and $d_{\mathrm{Y}}\left(\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1},\mathcal{T}\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1}\right)\to 0$ since $\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1}$ and $\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1}$ converge to the same $\widetilde{z}$, so due to their compatibility, we obtain $d_{\mathrm{Y}}\left(\mathcal{J}\mathcal{T}\widetilde{x}_{2\mathfrak{p}+1},\mathcal{T}\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1}\right)\to 0$ and, finally $\mathcal{T}\mathcal{J}\widetilde{x}_{2\mathfrak{p}+1}\to\mathcal{T}\widetilde{z}$. So, $\mathcal{J}\widetilde{z} = \mathcal{T}\widetilde{z}$.

Similarly, we have $\mathcal{I}\widetilde{z} = \mathcal{S}\widetilde{z}$. Indeed,

$$\begin{aligned}d_{\mathrm{Y}}\left(\mathcal{I}\widetilde{z},\mathcal{S}\widetilde{z}\right) &\leq d_{\mathrm{Y}}\left(\mathcal{I}\widetilde{z},\mathcal{I}\mathcal{S}\widetilde{x}_{2\mathfrak{p}}\right) + d_{\mathrm{Y}}\left(\mathcal{I}\mathcal{S}\widetilde{x}_{2\mathfrak{p}},\mathcal{S}\widetilde{z}\right)\\ &\leq d_{\mathrm{Y}}\left(\mathcal{I}\widetilde{z},\mathcal{I}\mathcal{S}\widetilde{x}_{2\mathfrak{p}}\right) + d_{\mathrm{Y}}\left(\mathcal{I}\mathcal{S}\widetilde{x}_{2\mathfrak{p}},\mathcal{S}\mathcal{I}\widetilde{x}_{2\mathfrak{p}}\right) + d_{\mathrm{Y}}\left(\mathcal{S}\mathcal{I}\widetilde{x}_{2\mathfrak{p}},\mathcal{S}\widetilde{z}\right)\\ &\to 0+0+0 = 0.\end{aligned}$$

If $\mathcal{S}\widetilde{z} \neq \mathcal{T}\widetilde{z}$ from (6) we obtain

$$\tau + \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{z}\right)\right) \leq \mathcal{F}\left(\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}\left(\widetilde{z}, \widetilde{z}\right)\right), \tag{15}$$

where

$$\begin{aligned}\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}\left(\widetilde{z}, \widetilde{z}\right) &= \max\left\{d_{\Upsilon}\left(\mathcal{I}\widetilde{z}, \mathcal{J}\widetilde{z}\right), d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{I}\widetilde{z}\right), d_{\Upsilon}\left(\mathcal{T}\widetilde{z}, \mathcal{J}\widetilde{z}\right), \frac{d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{J}\widetilde{z}\right) + d_{\Upsilon}\left(\mathcal{T}\widetilde{z}, \mathcal{I}\widetilde{z}\right)}{2}\right\} \\ &= \max\left\{d_{\Upsilon}\left(\mathcal{I}\widetilde{z}, \mathcal{J}\widetilde{z}\right), 0, 0, \frac{d_{\Upsilon}\left(\mathcal{I}\widetilde{z}, \mathcal{J}\widetilde{z}\right) + d_{\Upsilon}\left(\mathcal{J}\widetilde{z}, \mathcal{I}\widetilde{z}\right)}{2}\right\} \\ &= d_{\Upsilon}\left(\mathcal{I}\widetilde{z}, \mathcal{J}\widetilde{z}\right) = d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{z}\right).\end{aligned}$$

Now, (15) can be written in the form

$$\tau + \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{z}\right)\right) \leq \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{z}\right)\right),$$

which is a contradiction. Therefore, $\mathcal{S}\widetilde{z} = \mathcal{T}\widetilde{z}$. This further entails equality $\mathcal{I}\widetilde{z} = \mathcal{S}\widetilde{z} = \mathcal{J}\widetilde{z} = \mathcal{T}\widetilde{z}$. Let $\widetilde{w} = \mathcal{I}\widetilde{z} = \mathcal{S}\widetilde{z} = \mathcal{J}\widetilde{z} = \mathcal{T}\widetilde{z}$. Then we get

$$\mathcal{S}\widetilde{w} = \mathcal{S}\mathcal{I}\widetilde{z} = \mathcal{I}\mathcal{S}\widetilde{z} = \mathcal{I}\widetilde{w} \tag{16}$$

and

$$\mathcal{T}\widetilde{w} = \mathcal{T}\mathcal{J}\widetilde{z} = \mathcal{J}\mathcal{T}\widetilde{z} = \mathcal{J}\widetilde{w}. \tag{17}$$

If $\mathcal{S}\widetilde{z} \neq \mathcal{T}\widetilde{w}$ from (6) it follows

$$\tau + \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{w}\right)\right) \leq \mathcal{F}\left(\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}\left(\widetilde{z}, \widetilde{w}\right)\right), \tag{18}$$

where

$$\begin{aligned}\mathfrak{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{I},\mathcal{J}}\left(\widetilde{z}, \widetilde{w}\right) &= \max\left\{d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{w}\right), d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{S}\widetilde{z}\right), d_{\Upsilon}\left(\mathcal{T}\widetilde{w}, \mathcal{T}\widetilde{w}\right), \frac{d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{w}\right) + d_{\Upsilon}\left(\mathcal{T}\widetilde{w}, \mathcal{S}\widetilde{z}\right)}{2}\right\} \\ &= d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{w}\right),\end{aligned}$$

and now (18) can be written as

$$\tau + \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{w}\right)\right) \leq \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{z}, \mathcal{T}\widetilde{w}\right)\right), \tag{19}$$

which is a contradiction. Therefore, it must be $\mathcal{S}\widetilde{z} = \mathcal{T}\widetilde{w}$. Hence, $\mathcal{T}\widetilde{w} = \widetilde{w}$ and from (17) it follows that $\widetilde{w}$ is a common fixed point for $\mathcal{T}$ and $\mathcal{J}$. Similarly as in previous case, assumption $\mathcal{S}\widetilde{w} \neq \mathcal{T}\widetilde{z}$ implies a contradiction, since from (6) we get

$$\tau + \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{w}, \mathcal{T}\widetilde{z}\right)\right) \leq \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{S}\widetilde{w}, \mathcal{T}\widetilde{z}\right)\right).$$

Therefore, $\mathcal{S}\widetilde{w} = \mathcal{T}\widetilde{z}$. Suppose, further, that $\mathcal{S}\widetilde{w} = \widetilde{w}$. Then from (16) it follows that $\widetilde{w}$ is a common fixed point for $\mathcal{S}$ and $\mathcal{J}$. We proved that $\widetilde{w}$ is unique common fixed point for $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$. □

It is worth to notice that Theorem 5 generalizes Theorems 1 and 2 in several directions. Namely, putting $\mathcal{J} = \mathcal{I}$ and $\mathcal{S} = \mathcal{T}$ in (6) we get the following Jungck–Wardowski type result:

Theorem 6. *Let $(\mathcal{T}, \mathcal{I})$ be a pair of compatible self-mappings of a complete metric space (Υ, d_{Υ}) into itself and $\mathcal{F}\colon (0, +\infty) \to (-\infty, +\infty)$ is strictly increasing mapping such that*

$$\tau + \mathcal{F}\left(d_{\Upsilon}\left(\mathcal{T}\widetilde{x}, \mathcal{T}\widetilde{y}\right)\right) \leq \mathcal{F}\left(\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}\left(\widetilde{x}, \widetilde{y}\right)\right), \tag{20}$$

for all $\widetilde{x}, \widetilde{y} \in \Upsilon$ *with* $d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{T}\widetilde{y}) > 0$, *where*

$$\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y}) = \max\left\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{I}\widetilde{y}), d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{I}\widetilde{x}), d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{y}), \frac{d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{I}\widetilde{y}) + d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{x})}{2}\right\},$$

τ *is a given positive constant. If* $\mathcal{T}$ *and* $\mathcal{I}$ *are continuous and* $\mathcal{T}(\Upsilon) \subseteq \mathcal{I}(\Upsilon)$ *then* $\mathcal{T}, \mathcal{I}$ *have a unique common fixed point.*

Remark 3. *Replacing* $\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})$ *with*

$$\max\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{I}\widetilde{y}), d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{I}\widetilde{x}), d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{y})\},$$

or

$$\max\left\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{I}\widetilde{y}), \frac{d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{I}\widetilde{x}) + d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{y})}{2}, \frac{d_\Upsilon(\mathcal{T}\widetilde{x}, \mathcal{I}\widetilde{y}) + d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{x})}{2}\right\},$$

in (6) *we also find Theorem 6 to be true.*

As a result of Theorems 5 and 6 in the following we introduce new contractive conditions that complement the ones given in [11,27–30].

Corollary 1. *Suppose that* $(\mathcal{S}, \mathcal{I})$ *and* $(\mathcal{T}, \mathcal{J})$ *are the pairs of compatible self-mappings of a complete metric space* (Υ, d_Υ) *into itself such that for all* $\widetilde{x}, \widetilde{y} \in \Upsilon$ *with* $d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) > 0$ *there exist* $\tau_i > 0, i = \overline{1,7}$ *and the following inequalities hold true:*

$$\tau_1 + d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) \le \mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})$$

$$\tau_2 + \exp(d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y})) \le \exp\left(\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})\right)$$

$$\tau_3 - \frac{1}{d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y})} \le -\frac{1}{\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})}$$

$$\tau_4 - \frac{1}{d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y})} + d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) \le -\frac{1}{\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})} + \mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})$$

$$\tau_5 + \frac{1}{1 - \exp(d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}))} \le \frac{1}{1 - \exp(\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y}))}$$

$$\tau_6 + \frac{1}{\exp(-d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y})) - \exp(d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}))} \le \frac{1}{\exp(-\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})) - \exp(\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y}))}$$

$$\tau_7 + d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) \le d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y})$$

where $\mathfrak{M}_{\mathcal{T}}^{\mathcal{I}}(\widetilde{x}, \widetilde{y})$ *is one of the sets*

$$\max\left\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}), d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{I}\widetilde{x}), d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{J}\widetilde{y}), \frac{d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{J}\widetilde{y}) + d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{x})}{2}\right\},$$

$$\max\left\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}), \frac{d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{I}\widetilde{x}) + d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{J}\widetilde{y})}{2}, \frac{d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{J}\widetilde{y}) + d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}x)}{2}\right\}$$

or

$$\max\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y})\} = d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}).$$

If $\mathcal{I}, \mathcal{J}, \mathcal{S}$ *and* $\mathcal{T}$ *are continuous and* $\mathcal{S}(\Upsilon) \subseteq \mathcal{J}(\Upsilon), \mathcal{T}(\Upsilon) \subseteq \mathcal{I}(\Upsilon)$, *then in every of these cases mappings* $\mathcal{I}, \mathcal{J}, \mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point in* Υ.

Proof. Take, in Theorems 5 and 6 $\mathcal{F}(\mathfrak{c}) = \mathfrak{c}$, $\mathcal{F}(\mathfrak{c}) = \exp(\mathfrak{c})$, $\mathcal{F}(\mathfrak{c}) = -\frac{1}{\mathfrak{c}}$,$\mathcal{F}(\mathfrak{c}) = -\frac{1}{\mathfrak{c}} + \mathfrak{c}$, $\mathcal{F}(\mathfrak{c}) = \frac{1}{1-\exp(\mathfrak{c})}$, $\mathcal{F}(\mathfrak{c}) = \frac{1}{\exp(-\mathfrak{c})-\exp(\mathfrak{c})}$ respectively. Since every one of the functions $\mathfrak{c} \to \mathcal{F}(\mathfrak{c})$ is strictly increasing on $(0, +\infty)$ the result follows by Theorems 5 and 6. □

The next example supports Theorem 5. In fact, it is a modification of an example given in [31].

Example 1. *Let* $\Upsilon = [0,1]$ *and* $d_\Upsilon(\widetilde{x}, \widetilde{y}) = |\widetilde{x} - \widetilde{y}|$ *be a standard metric on it. Let us define the mappings* $\mathcal{I}, \mathcal{J}, \mathcal{S}, \mathcal{T} \colon \Upsilon \to \Upsilon$ *for all* $\widetilde{x} \in \Upsilon$ *as*

$$\mathcal{I}(\widetilde{x}) = \left(\frac{\widetilde{x}}{3}\right)^8, \mathcal{J}(\widetilde{x}) = \left(\frac{\widetilde{x}}{3}\right)^4, \mathcal{S}(\widetilde{x}) = \left(\frac{\widetilde{x}}{3}\right)^{16} \text{ and } \mathcal{T}(\widetilde{x}) = \left(\frac{\widetilde{x}}{3}\right)^8.$$

Obviously, $\mathcal{I}, \mathcal{J}, \mathcal{S}, \mathcal{T}$ *are self mappings and inclusions* $\mathcal{S}(\Upsilon) \subseteq \mathcal{J}(\Upsilon), \mathcal{T}(\Upsilon) \subseteq \mathcal{I}(\Upsilon)$ *are valid. Further, a pair* $(\mathcal{S}, \mathcal{I})$ *is compatible. Really, if* $\{\widetilde{x}_\mathfrak{p}\}$ *is a sequence in* Υ *such that*

$$\lim_{\mathfrak{p}\to+\infty} \mathcal{S}\widetilde{x}_\mathfrak{p} = \lim_{\mathfrak{p}\to+\infty} \mathcal{I}\widetilde{x}_\mathfrak{p} = \widetilde{x}, \text{ for some } \widetilde{x} \in \Upsilon,$$

then due to the continuity of $\mathcal{S}$ *and* $\mathcal{I}$ *it follows*

$$\begin{aligned}\lim_{\mathfrak{p}\to+\infty} d_\Upsilon\left(\mathcal{S}\left(\mathcal{I}\left(\widetilde{x}_\mathfrak{p}\right)\right), \mathcal{I}\left(\mathcal{S}\left(\widetilde{x}_\mathfrak{p}\right)\right)\right) &= \lim_{\mathfrak{p}\to+\infty} \left|\mathcal{S}\left(\mathcal{I}\left(\widetilde{x}_\mathfrak{p}\right)\right) - \mathcal{I}\left(\mathcal{S}\left(\widetilde{x}_\mathfrak{p}\right)\right)\right| \\ &= \left|\mathcal{S}(\widetilde{x}) - \mathcal{I}(\widetilde{x})\right| \\ &= \left|\left(\frac{\widetilde{x}}{3}\right)^{16} - \left(\frac{\widetilde{x}}{3}\right)^8\right| = \left(\frac{\widetilde{x}}{3}\right)^8 \left|\left(\frac{\widetilde{x}}{3}\right)^4 - 1\right| \left|\left(\frac{\widetilde{x}}{3}\right)^4 + 1\right| = 0,\end{aligned}$$

only for $\widetilde{x} = 0$. *Similarly we can prove that the second pair* $(\mathcal{T}, \mathcal{J})$ *is compatible. Furthermore, it is easy to show that both pairs are not commuting.*

For $\widetilde{x}, \widetilde{y} \in \Upsilon$, *we obtain*

$$\begin{aligned} d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{T}\widetilde{y}) &= |\mathcal{S}\widetilde{x} - \mathcal{T}\widetilde{y}| \\ &= \left|\left(\frac{\widetilde{x}}{3}\right)^{16} - \left(\frac{\widetilde{y}}{3}\right)^8\right| = \left(\left(\frac{\widetilde{x}}{3}\right)^8 + \left(\frac{\widetilde{y}}{3}\right)^4\right) \left|\left(\frac{\widetilde{x}}{3}\right)^8 - \left(\frac{\widetilde{y}}{3}\right)^4\right| \\ &\le \left(\frac{1}{6561} + \frac{1}{81}\right) d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}) = \frac{82}{6561} d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}).\end{aligned}$$

Putting $\tau = \ln \frac{6561}{82}$, $\mathcal{F}(\omega) = \ln \omega$ *we get that the condition* (6) *holds true. Based on Theorem* 5, *this means that the mappings* $\mathcal{I}, \mathcal{J}, \mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point* $\widetilde{x} = 0$.

Finaly, we believe that the following problem may be interesting for some future research :
Conjecture: Prove or disprove that Theorem 5 holds true if for the set $\mathfrak{M}^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{T}}(\widetilde{x}, \widetilde{y})$ we put

$$\mathfrak{M}^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{T}}(\widetilde{x}, \widetilde{y}) = \max\left\{d_\Upsilon(\mathcal{I}\widetilde{x}, \mathcal{J}\widetilde{y}), d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{I}\widetilde{x}), d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{J}\widetilde{y}), d_\Upsilon(\mathcal{S}\widetilde{x}, \mathcal{J}\widetilde{y}), d_\Upsilon(\mathcal{T}\widetilde{y}, \mathcal{I}\widetilde{x})\right\}.$$

Author Contributions: Conceptualization, S.R. (Stojan Radenović), J.V. and E.L.; methodology, J.V., S.R. (Slobodan Radojević) and E.L.; validation, S.R. (Stojan Radenović) and J.V.; investigation, S.R. (Stojan Radenović), J.V., E.L. and S.R. (Slobodan Radojević); All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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