



Realizing Brachistochronic Planar Motion of a Variable Mass Nonholonomic Mechanical System by an Ideal Holonomic Constraint with Restricted Reaction

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Abstract. The paper considers realization of the brachistochronic motion of a nonholonomic mechanical system, composed of variable mass particles, by means of an ideal holonomic constraint with restricted reaction. It is assumed that the system performs planar motion in an arbitrary field of forces and that it has two degrees of freedom. In addition, the laws of the time-rate of mass variation of the particles, as well as relative velocities of the expelled and gained particles, respectively, are known. Restricted reaction of the holonomic constraint is taken for the scalar control. Applying Pontryagin's maximum principle and singular optimal control theory, the problem of brachistochronic motion is solved as a two-point boundary value problem (TPBVP). Since the reaction of the constraint is restricted, different types of control structures are examined, from singular to totally nonsingular. The considerations are illustrated via an example.

1. Introduction

Generalizations of the classical brachistochrone problem of a material point in a vertical plane, whose motion is realized by an ideal constraint without active control forces, are still topical today. This is confirmed by a doctoral dissertation defended recently at the Lomonosov Moscow State University [1]. Throughout literature it is possible to encounter works related to the brachistochronic motion of a material point, of both constant and variable mass [2–7], as well as works related to the brachistochronic motion of mechanical systems [8–15]. Regarding variable mass nonholonomic mechanical systems, there is not a lot of works on that subject, whether it is the application of other types of equations in that field, such as Kane's [16] or Hamilton's equations [17], or the control of such systems [8–14, 18]. Pontryagin's maximum principle [19–22], as well as the optimal control theory [19–21] can be applied in solving the brachistochrone problem. Considerations in this work rely on [8]. The aim of the work is to realize the brachistochronic motion of a variable mass nonholonomic mechanical system with the help of an ideal holonomic constraint with restricted reaction, and therefore it represents a kind of continuation of work [8]. To the best of

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the authors' knowledge, such problem has not been considered yet. The work is organized as follows: in Section 2 the brachistochrone problem of a variable mass nonholonomic mechanical system with two degrees of freedom is defined; in Section 3 the brachistochrone problem is formulated as the task of optimal control, which can be solved by scalar control, and numerical procedure for solving the obtained TPBVP based on the shooting method is presented; to show the obtained results, Section 4 gives an example of the mechanical system, which is a modification of the example from [9]; conclusion is contained in Section 5.

2. Formulation of the problem

Consider planar motion of the mechanical system composed of N material points. Without loss of generality all material points can be of variable mass. The system configuration is defined by means of n generalized coordinates $q=(q^1, q^2, \dots, q^n)^T$, which are geometrically independent, and based on them the mechanical system position is unambiguously determined. In addition, the laws of the time-rate of mass variation of the material points can be considered to be known:

$$m_l = m_l(t), \quad l = 1, \dots, N, \quad (1)$$

where $m_l(t)$ are continuous and differentiable functions of time. Mass variation can be realized by expelling or gaining of masses, assuming that the process of expelling and gaining of masses, respectively, is continuous over the considered interval of time. Relative velocities of expelling and gaining of masses, respectively, are considered to be known:

$$\vec{v}_l^{rel} = \vec{v}_l^{rel}(q, \dot{q}, t), \quad l = 1, \dots, N, \quad (2)$$

where $\dot{q}=(\dot{q}^1, \dot{q}^2, \dots, \dot{q}^n)^T$ is the vector of generalized velocities. The kinetic energy of a scleronomic mechanical system is a homogeneous quadratic form of generalized velocities [23, 24]:

$$T = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j, \quad i, j = 1, \dots, n, \quad (3)$$

where $a_{ij} = a_{ij}(q, t)$ are the covariant coordinates of metric tensor of the function of generalized coordinates and time t . At the same time, the existence of variable mass material points should be taken into account (1). Also, the well known Einstein summation convention is deployed in the paper, where the indices have a range of values as follows: $i, j, k, r = 1, \dots, n; \alpha, \beta, \gamma, \delta = 1, 2; \nu, \rho = 3, \dots, n$. Planar motion of the considered mechanical system is constrained by p ideal independent stationary nonholonomic homogeneous constraints of the form:

$$\gamma^\nu(q, \dot{q}) \equiv \dot{q}^\nu - c_\alpha^\nu \dot{q}^\alpha = 0, \quad (4)$$

where $c_\alpha^\nu = c_\alpha^\nu(q)$. Number p is taken in such way that the number of degrees of freedom of a mechanical system motion is $m=n-p=2$, and therefore $p=n-2$. At the same time, $m=2$ represents the number of kinematically independent coordinates q^α , which correspond to independent generalized velocities \dot{q}^α that can be expressed as a linear form of independent quasi-velocities V^β [23, 24]:

$$\dot{q}^\alpha = b_\beta^\alpha V^\beta. \quad (5)$$

If (4) and (5) are taken into account, dependent generalized velocities can be written as follows:

$$\dot{q}^\nu = b_\beta^\nu V^\beta, \quad (6)$$

where $b_\beta^\nu = c_\alpha^\nu b_\beta^\alpha$. Based on expressions (5) and (6), the transformations of all generalized velocities, both dependent and independent, can be expressed as the linear forms of independent quasi-velocities:

$$\dot{q}^i = b_\alpha^i V^\alpha, \quad (7)$$

where $b_\alpha^i = b_\alpha^i(\mathbf{q})$ are continuous functions with continuous first derivatives in the area of mechanical system considerations. Using previous expression (7) and in accordance with (3), kinetic energy of the nonholonomic scleronomic mechanical system can be also expressed as a homogeneous quadratic form of independent quasi-velocities:

$$T^* = \frac{1}{2} G_{\alpha\beta} V^\alpha V^\beta, \tag{8}$$

where:

$$G_{\alpha\beta}(\mathbf{q}, t) = a_{ij} b_\alpha^i b_\beta^j, \tag{9}$$

and where $G_{\alpha\beta}$ are the covariant coordinates of metric tensor relative to kinematically independent coordinates q^α taking into account independent quasi-velocities V^α . It can be considered that the studied mechanical system is moving in a field of known potential forces, whose potential energy equals:

$$\Pi = \Pi(\mathbf{q}, t), \tag{10}$$

and that the system is acted on by known arbitrary nonpotential forces, so that the generalized forces are:

$$Q_i^w = Q_i^w(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{11}$$

In order to create differential equations of motion as a function of kinematically independent coordinates, we will start from Lagrange-D'Alembert's principle [23, 24]:

$$(a_{ij} a^j - Q_i) \delta q^i = 0, \tag{12}$$

where Q_i are covariant generalized forces corresponding to geometrically independent coordinates, and a^j are the contravariant coordinates of acceleration that can be written as follows:

$$a^j = \ddot{q}^j + \Gamma_{kr}^j \dot{q}^k \dot{q}^r, \tag{13}$$

where Γ_{kr}^j are Christoffel symbols of the second kind. Applying the Hertz-Hedler principle [25, 26], and according to (7), it can be written:

$$\delta q^i = b_\alpha^i \delta \pi^\alpha, \tag{14}$$

where $\delta \pi^\alpha$ are variations of independent quasi-coordinates, where $\dot{\pi}^\alpha = V^\alpha$ holds. Due to variations' independence, that is $\delta \pi^\alpha \neq 0$, and taking into account (7), (12), (13) and (14), and using the contravariant coordinates of metric tensor $G^{\alpha\beta}$, after rearrangement the differential equations of motion for the considered system are obtained [8]:

$$\dot{V}^\beta = G^{\alpha\beta} \Delta_\alpha, \tag{15}$$

where:

$$\Delta_\alpha(\mathbf{q}, \mathbf{V}, t) = \tilde{Q}_\alpha - a_{ij} b_\alpha^i b_\gamma^k \left(\frac{\partial b_\beta^j}{\partial q^k} + \Gamma_{kr}^j b_\beta^r \right) V^\gamma V^\beta, \tag{16}$$

whereas the generalized forces corresponding to kinematically independent coordinates are represented as:

$$\tilde{Q}_\alpha(\mathbf{q}, \mathbf{V}, t) = b_\alpha^i Q_i, \tag{17}$$

where $V = (V^1, V^2)^T$. The generalized forces corresponding to geometrically independent coordinates can be represented, in a general case, in the form as follows [27, 28]:

$$Q_i(q, \dot{q}, t) = -\frac{\partial \Pi}{\partial \dot{q}^i} + Q_i^w + Q_i^{var} + Q_i^c + Q_i^\Lambda. \tag{18}$$

The generalized reaction forces that develop due to expelling and gaining of masses, respectively, can be written as [27, 28]:

$$Q_i^{var}(q, \dot{q}, t) = \sum_{l=1}^N m_l \vec{v}_l^{rel} \cdot \frac{\partial \vec{r}_l}{\partial \dot{q}^i}, \tag{19}$$

while at the same time $Q_i^c = Q_i^c(q, \dot{q}, t)$ are generalized control forces, whose total power during brachistochronic motion equals zero:

$$Q_i^c \dot{q}^i = 0, \tag{20}$$

where, in accordance with (7) and (17) it can be written:

$$Q_\alpha^c \dot{V}^\alpha = 0. \tag{21}$$

Since generalized forces due to imposed nonholonomic constraints (4) can be written in the form as follows:

$$Q_i^\Lambda(q, \dot{q}) = \Lambda_\nu \frac{\partial \gamma^\nu}{\partial \dot{q}^i}, \tag{22}$$

where Λ_ν are Lagrange’s multipliers of the constraints, based on (4), (17) and (22), it can be shown that:

$$\tilde{Q}_\alpha^\Lambda = b_\alpha^\beta Q_\beta^\Lambda + b_\alpha^\nu Q_\nu^\Lambda = \Lambda_\nu (b_\alpha^\nu - b_\alpha^\beta c_\beta^\nu) = 0. \tag{23}$$

Based on these equations, it can be concluded that Lagrange’s multipliers of the constraints do not occur in differential equations of motion (15), and hence the procedure of defining the reactions of nonholonomic constraints is completely separated from the procedure of defining the system motion.

The question is posed on realizing the motion of the presented mechanical system. The answer is found in subsequently imposed ideal holonomic constraint. Since it is the mechanical system with two kinematically independent generalized coordinates, the motion can be realized by the imposition of smooth guides to a single material point, whose motion is defined by previous numerical integration of differential equations. Without loss of generality, let it be the point S of one body of the system. This way, the brachistochronic motion is realized without active forces’ influence, which is in accordance with the elementary brachistochrone problem of a material point in a vertical plane.

Let the values of generalized coordinates be specified, as well as the value of mechanical energy of the mechanical system at the initial instant of time:

$$t_0 = 0, \quad q(t_0) = q_0, \tag{24}$$

$$T^*(q_0, V_0, t_0) + \Pi(q_0, t_0) = E_0, \tag{25}$$

and also the values of generalized coordinates corresponding to the final position of the system:

$$q(t_f) = q_f, \tag{26}$$

where $E_0 \in \mathbb{R}$ and $t_f \in \mathbb{R}$. The problem of brachistochronic planar motion of a variable mass nonholonomic mechanical system, whose differential equations of motion are given in the form (15), consists of defining the generalized control forces $Q_i^c = Q_i^c(t)$, which are reduced in this case to defining the reaction of imposed ideal holonomic constraint that can be restricted in this case and corresponding equations of the system motion $q_i = q_i(t)$, so that the system moves in the minimum time t_f from the initial state defined by (24) and (25) to the final position defined by (26).

3. Brachistochrone problem as an optimal control task

The presented brachistochrone problem can be formulated as a task of optimal control by introducing scalar control u :

$$u = R_S, \tag{27}$$

where R_S is the projection of the imposed ideal holonomic constraint at point S of the mechanical system. In that case, the constraint reaction vector, taking into account that the total power of control forces on the brachistochronic motion equals zero, is defined by:

$$\vec{R}_S = u \frac{\vec{v}'_S}{|\vec{v}'_S|} = u \frac{-\dot{y}_S \vec{i} + \dot{x}_S \vec{j}}{\sqrt{\dot{x}_S^2 + \dot{y}_S^2}}, \tag{28}$$

where \vec{v}_S the velocity of the point S , \vec{v}'_S is such a vector that it is fulfilled $|\vec{v}_S| = |\vec{v}'_S|$ and $\vec{v}_S \cdot \vec{v}'_S = 0$. In order to define the generalized control forces, it is needed to define the elementary work of the constraint reaction:

$$\delta A(\vec{R}_S) = \vec{R}_S \cdot \delta \vec{r}_S, \tag{29}$$

where $\delta \vec{r}_S = \delta x_S \vec{i} + \delta y_S \vec{j}$ is variation of the position vector of material point S . Having in mind that this is a nonholonomic scleronomic system, the following expressions hold [23]:

$$\begin{aligned} \delta x_S &= \frac{\partial x_S}{\partial q^i} \delta q^i, \\ \delta y_S &= \frac{\partial y_S}{\partial q^i} \delta q^i, \\ \dot{x}_S &= \frac{\partial x_S}{\partial q^i} \dot{q}^i, \\ \dot{y}_S &= \frac{\partial y_S}{\partial q^i} \dot{q}^i. \end{aligned} \tag{30}$$

Now, based on (28)), (29) and (30) the generalized control forces can be expressed:

$$Q_i^c = u \frac{e^i_j \dot{q}^j}{\sqrt{\left(\frac{\partial x_S}{\partial q^i} \dot{q}^i\right)^2 + \left(\frac{\partial y_S}{\partial q^i} \dot{q}^i\right)^2}}, \tag{31}$$

where $e^i_j = \frac{\partial x_S}{\partial q^j} \frac{\partial y_S}{\partial q^i} - \frac{\partial y_S}{\partial q^j} \frac{\partial x_S}{\partial q^i}$. Taking into account (17) it follows:

$$\tilde{Q}_\alpha^c = D_i b^i_\beta u = g_\beta(\mathbf{q}, \mathbf{V})u. \tag{32}$$

The normal form of first-order differential equations, known in the optimal control theory as the state equations, can be written by incorporating the rheonomic coordinate $q^{n+1} \triangleq t$ in the following manner:

$$\begin{aligned} \dot{q}^i &= f_i(\mathbf{q}, \mathbf{V}, q^{n+1}, u) \equiv b^i_\alpha V^\alpha, \\ \dot{q}^{n+1} &= f_{n+1}(\mathbf{q}, \mathbf{V}, q^{n+1}, u) \equiv 1, \\ \dot{V}^\alpha &= f_\alpha(\mathbf{q}, \mathbf{V}, q^{n+1}, u) \equiv c^\alpha(\mathbf{q}, \mathbf{V}, q^{n+1}) + d^\alpha(\mathbf{q}, \mathbf{V}, q^{n+1})u, \end{aligned} \tag{33}$$

where:

$$\begin{aligned} c^\alpha &= G^{\alpha\beta}(\tilde{Q}_\alpha^\Pi + \tilde{Q}_\alpha^w \tilde{Q}_\alpha^{var} - a_{ij} b_\alpha^i b_\gamma^j (\frac{\partial b_\beta^j}{\partial q^k} + \Gamma_{kr}^j b_\beta^r) V^\gamma V^\beta), \\ d^\alpha &= G^{\alpha\beta} g_\beta. \end{aligned} \tag{34}$$

The brachistochrone problem of the considered nonholonomic system motion described by the state equations (34), consists of defining the optimal scalar control u and corresponding optimal trajectories in state space $q^i(t)$, so that the mechanical system moves from the initial state defined by (24) and (25) to the final position (26), in the minimum time, which can be expressed using conditions for the functional [19]:

$$J(q, V, q^{n+1}, u) = \int_0^{t_f} dt, \tag{35}$$

over the interval $[0, t_f]$ it has minimum value. In order to solve the problem of optimal control by applying Pontryagin’s maximum principle [21], the Hamiltonian is created of the Hamilton-Pontryagin form:

$$H(q, V, q^{n+1}, u, \lambda, v) = -1 + \lambda_j b_\alpha^j V^\alpha + \lambda_{n+1} + v_\alpha (c^\alpha + d^\alpha u), \tag{36}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})^T$, $v = (v_1, v_2)^T$, whereas $\lambda_i(\cdot) : [0, t_f] \rightarrow \mathbb{R}$, $\lambda_{n+1}(\cdot) : [0, t_f] \rightarrow \mathbb{R}$ and $v_\alpha(\cdot) : [0, t_f] \rightarrow \mathbb{R}$ are costate variables, so that the costate system of differential equations has the form:

$$\begin{aligned} \dot{\lambda}_i &= -\frac{\partial H}{\partial q^i} = -\lambda_j \frac{\partial b_\alpha^j}{\partial q^i} V^\alpha - v_\alpha \left(\frac{\partial c^\alpha}{\partial q^i} + \frac{\partial d^\alpha}{\partial q^i} u \right), \\ \dot{\lambda}_{n+1} &= -\frac{\partial H}{\partial q^{n+1}} = -\lambda_j \frac{\partial b_\alpha^j}{\partial q^{n+1}} V^\alpha - v_\alpha \left(\frac{\partial c^\alpha}{\partial q^{n+1}} + \frac{\partial d^\alpha}{\partial q^{n+1}} u \right), \\ \dot{v}_\beta &= -\frac{\partial H}{\partial V^\beta} = -\lambda_j b_\beta^j - v_\alpha \left(\frac{\partial c^\alpha}{\partial V^\beta} + \frac{\partial d^\alpha}{\partial V^\beta} u \right). \end{aligned} \tag{37}$$

Based on (36), it can be written:

$$H(q, V, q^{n+1}, u, \lambda, v) = H_0 + H_1 u, \tag{38}$$

where:

$$\begin{aligned} H_0 &= -1 + \lambda_j b_\alpha^j V^\alpha + \lambda_{n+1} + v_\alpha c^\alpha, \\ H_1 &= v_\alpha d^\alpha. \end{aligned} \tag{39}$$

In a general case, the reaction of an ideal holonomic constraint can be restricted, and thus the control is restricted too:

$$|u| \leq C \tag{40}$$

where C is a restriction of the constraint reaction. Since scalar control figures linearly in the state equations, it is needed to consider the possibility of singular solutions occurrence in solving the TPBVP. Singular solutions, depending on the capability of constraints (permitted intensity of constraint reaction), may occur over the entire interval or over singular parts. Consequently, the control can take the following forms:

$$u = \begin{cases} u_{sing}, & H_1 = 0, \\ C * \text{sgn}H_1, & H_1 \neq 0, \end{cases} \tag{41}$$

where $\text{sgn}()$ is the signum function. So, it is necessary to simultaneously solve the systems (33) and (37), where depending on boundary (40) a singular solution may occur over the entire interval, a combination of singular and nonsingular solutions, and a bang-bang type of solution. For the case of control known in

the optimal control theory as a singular control, whether it occurs over a part or over the entire interval, the necessary optimality condition of Pontryagin’s maximum principle is of the form as follows [20]:

$$\frac{\partial H}{\partial u} = H_1 = 0, \tag{42}$$

from where singular optimal control u cannot be explicitly defined. Hence, it is required that H_1 be identically equal to zero alongside the optimal trajectory of state. Singular optimal control u is defined by further differentiation with respect to time (42) taking into account (33) and (37):

$$\frac{d^k}{dt^k} \left[\frac{\partial H}{\partial u} \right] = 0, \quad k = 1, 2, \dots \tag{43}$$

In defining the relations (43) the Poisson bracket formalism will be applied [29]:

$$\dot{H}_1 = \{H, H_1\} = \{H_1, H_0\} + \{H_1, H_1\}u = 0. \tag{44}$$

Taking into account (42), as well as that $\{H_1, H_1\} = 0$ [29], it is obtained:

$$\{H_1, H_0\} = \sum_{\theta=1}^{n+3} \left(\frac{\partial H_1}{\partial y^\theta} \frac{\partial H_0}{\partial \zeta^\theta} - \frac{\partial H_1}{\partial \zeta^\theta} \frac{\partial H_0}{\partial y^\theta} \right) = 0, \tag{45}$$

where $\mathbf{y} = (y^1, y^2, \dots, y^{n+3}) \triangleq (q^1, q^2, \dots, q^{n+1}, V^1, V^2)$ and $\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \dots, \zeta^{n+3}) \triangleq (\lambda^1, \lambda^2, \dots, \lambda^{n+1}, \nu^1, \nu^2)$. Further differentiation (45) yields:

$$\left\{ \{H_1, H_0\}, H_0 \right\} + \left\{ \{H_1, H_0\}, H_1 \right\} u = 0. \tag{46}$$

From where singular control can be expressed as:

$$u = - \frac{\left\{ \{H_1, H_0\}, H_0 \right\}}{\left\{ \{H_1, H_0\}, H_1 \right\}}. \tag{47}$$

Furthermore, the transversality conditions can be represented in the form as follows:

$$\left(\lambda_i \Delta q^i + \lambda_{n+1} \Delta q^{n+1} + \nu_\alpha \Delta V^\alpha \right) \Big|_0^{t_f} = 0, \tag{48}$$

$$(H \Delta t) \Big|_0^{t_f} = 0, \tag{49}$$

where $\Delta(\cdot)$ is asynchronous variation [23, 24] of the quantity (\cdot) . Based on condition (42), the costate variable ν_1 can be expressed as a function of the costate variable ν_2 :

$$\nu_1 = \nu_2 \frac{d^2}{d^1}. \tag{50}$$

Now, from equations (45), taking into account (39) and (50), one can express:

$$\lambda_1(\mathbf{q}, \mathbf{V}, q^{n+1}, \lambda_2, \lambda_3, \dots, \lambda_n, \nu_2) = \frac{1}{b_\alpha^1 d^\alpha} (\nu_\alpha \phi - \lambda_2 b_\alpha^2 d^\alpha - \lambda_\rho b_\alpha^\rho d^\alpha), \tag{51}$$

where:

$$\phi = \frac{\partial d^\alpha}{\partial q^i} b_i^\alpha V^\alpha + \frac{\partial d^\alpha}{\partial q^{n+1}} + \frac{\partial d^\alpha}{\partial V^\beta} c^\alpha - \frac{\partial c^\alpha}{\partial V^\beta} d^\alpha. \tag{52}$$

Since the initial position of the mechanical system according to (24) is defined, it follows:

$$\Delta t(t_0) = 0, \quad \Delta q^i(t_0) = 0, \quad \Delta q^{n+1}(t_0) = 0. \tag{53}$$

If (53) is taken into account and the operator of asynchronous variation is applied to (25), it can be obtained:

$$G_{\alpha\beta}(t_0)V^\beta(t_0)\Delta V^\alpha(t_0) = 0, \tag{54}$$

and lastly, after substituting (50) and (53) into (48), it is obtained:

$$v_\alpha(t_0)\Delta V^\alpha(t_0) = v_2(t_0)G_{\alpha\beta}(t_0)V^\beta(t_0)\Delta V^\alpha(t_0) = 0. \tag{55}$$

Based on (53), (54) and (55), it is obvious that the transversality conditions (48) and (49) in the initial configuration of the system are satisfied. In the final configuration (26) of the mechanical system the time is not known, and based on it, the transversality condition results from (49):

$$H(t_f) = 0, \tag{56}$$

and as quantities $V^\alpha(t_f)$ and $q^{n+1}(t_f)$ are not a priori defined ($\Delta V^\alpha(t_f) \neq 0, \Delta q^{n+1}(t_f) \neq 0$), the next transversality conditions are obtained from (48):

$$v_\alpha(t_f) = 0, \lambda_{n+1}(t_f) = 0. \tag{57}$$

Based on (36), (51), (56) and (57), the following dependence can be established in analytical form:

$$\lambda_e(t_f) = \lambda_e(t_f)(\mathbf{V}(t_f), q^{n+1}(t_f), \lambda_{i,i \neq 1, i \neq e}(t_f)), \tag{58}$$

where $\mathbf{V}(t_f) = (V^1(t_f), V^2(t_f))^T$ and $q^{n+1}(t_f) = t_f$.

If considerations are restricted to the first order singular controls, where $\{\{H_1, H_0\}, H_1\} \neq 0$, using (50) and (51), singular scalar control u_{sing} from (47) can be represented in the form as follows:

$$u_{sing} = u_{sing}(q, \mathbf{V}, q^{n+1}, \lambda_2, \lambda_3, \dots, \lambda_n, v_2). \tag{59}$$

Also, the Kelley necessary condition for the first order singular control is given in the form:

$$-\frac{\partial}{\partial u} \left(\frac{d^2}{dt^2} \left[\frac{\partial H}{\partial u} \right] \right) \leq 0, \tag{60}$$

Applying the Poisson brackets, this condition is reduced to:

$$K = \{\{H_1, H_0\}, H_1\} > 0. \tag{61}$$

For the case of nonsingular control (42) does not hold, i.e. $H_1 \neq 0$, so that control over the part or over the entire interval (in the case of bang-bang control) has the value $C \cdot \text{sgn} H_1$. Since (42) does not hold, it follows that (43)-(47) do not hold either, and therefore nor can (51) be expressed, so that $\lambda_1(t_f)$ also figures in the expression (58). Also, since (42) does not hold, nor does the transversality condition at the start of the interval (55) hold, based on (48) and (54)), it can only be written:

$$(v_\alpha(t_0) - G_{\alpha\beta}(t_0)V^\beta(t_0))\Delta V^\alpha(t_0) = 0, \tag{62}$$

and since $V^\alpha(t_0)$ is not a priori defined ($\Delta V^\alpha(t_0) \neq 0$), it follows:

$$v_\alpha(t_0) - G_{\alpha\beta}(t_0)V^\beta(t_0) = 0. \quad (63)$$

Note that for the case of existing singular and nonsingular parts of control, the junction conditions should be satisfied. The corresponding conditions for the junction between a singular and nonsingular part of extremal control, representing necessary conditions for the optimal junction, must be satisfied, as defined by Theorem 1 from [30, 31]. Namely, if $2q$ is time derivative bottom row of a discontinuous function H_1 , which contains explicit control u , and $u^{(r)}(\geq 0)$ is derivative bottom row of control u that has a disruption at the junction moment t_1 , in accordance with Theorem 1 [30, 31], the necessary junction condition between a singular and nonsingular part of the extremal control is expressed by the condition that the sum $q+r$ is an odd whole number. If $q=1$ (first order control) and $r=0$ ($u(t)$ has a disruption at the junction point), conditions are satisfied, which will be the case in the herein considered example.

Also, it should be noted that as the boundary decreases (40) the structure of control changes. Hence, the example will be an attempt to explain how different control structures can be obtained for different boundary values (40).

Substituting (41) into (33) and (37) yields a two-point boundary value problem (TPBVP) with $2n+6$ first-order nonlinear normal form differential equations. Due to nonlinearity, in a general case, it is necessary to apply the appropriate numerical method [32]. In this paper, the shooting method will be deployed.

In case that restriction is not imposed to control (40), it is singular over the entire interval $[0, t_f]$. The shooting method is most suitable to perform in this case by the backward numerical integration choosing the $(n+1)$ values $\lambda_{i,i \neq 1, i \neq e}(t_f), V^\alpha(t_f), t_f$, which will ensure fulfillment of the same number of initial conditions (24) and (25). The value $\lambda_1(t_f)$ was defined via (51) for $t=t_f$, and $\lambda_e(t_f)$ from the expression (58)).

If there is a restriction (40)), it is also necessary to consider the occurrence of the parts of extremal trajectory, where $u(t)=C*\text{sgn}H_1$, and in this case the first step in solving the problem is to define a singular solution. If $u^* = u_{\text{sing,max}}(t) \leq C$, then singular control is extremal. If this is not the case, it should be examined what happens with the control structure by further boundary decrease below u^* , until the fulfillment of (40). In seeking a solution it will be assumed that parts over which $u(t)=C\text{sgn}H_1$ are where singular control over the entire interval has extended beyond the boundary. It can happen at the ends of the interval or somewhere in the middle.

If the structure of control is such that $u(t)=C*\text{sgn}H_1$ at the end of the interval $[0, t_f]$, then the shooting method consists of choosing $(n+1)$ values $\lambda_{i,i \neq 1, i \neq e}(t_f), V^\alpha(t_f), t_f$ which will ensure fulfillment of the same number of initial conditions (24), (25) and (63).

At junction points, corresponding to time moments t^* , the conditions (42) and (44) are satisfied, whereas Kelley's conditions (61) should be checked on the singular parts.

Numerical solution of the problem can be performed applying a software package *Wolfram Mathematica* [33] in two steps. In step 1 numerical relations are established in the form of the system of differential equations with unknown values that are chosen. In establishing these relations the functions *NDSolve[]* and *First[]* are employed. In step 2 unknown boundary values are defined by applying the function *FindRoot[]*. After the appropriate boundary values are defined, the system of differential equations is solved by applying the function *NDSolve[]*. Thus, the given problem is solved and will be presented using an example.

4. Numerical example

The example shows a nonholonomic mechanical system composed of two variable mass material points A and B with an imposed constraint of motion in the form of perpendicularity of the velocities by means

of Chaplygin sleighs of negligible masses, as indicated by Fig. 1. In step 1, for the needs of further considerations, two Cartesian coordinate reference systems must be introduced. The first, a stationary coordinate system $Oxyz$, whose coordinate plane Oxy coincides with the horizontal plane of motion, and the second, a non-stationary coordinate system $B\xi\eta\zeta$, whose coordinate origin is attached to material point B of the system, the coordinate plane $B\xi\eta$ coinciding with the plane Oxy . In addition, the axis of the non-stationary coordinate system $B\xi$ is defined by the direction BA , that is, $A \in B\xi$. Unit vectors of the non-stationary coordinate system axes are $\vec{\lambda}, \vec{\mu}, \vec{\nu}$ and , respectively. Variable-mass material points A and B are interconnected by a lightweight mechanism of the ‘pitchfork’ type, which allows the distance between the points to change, i.e. $\overline{BA} = \xi \neq \text{const.}$

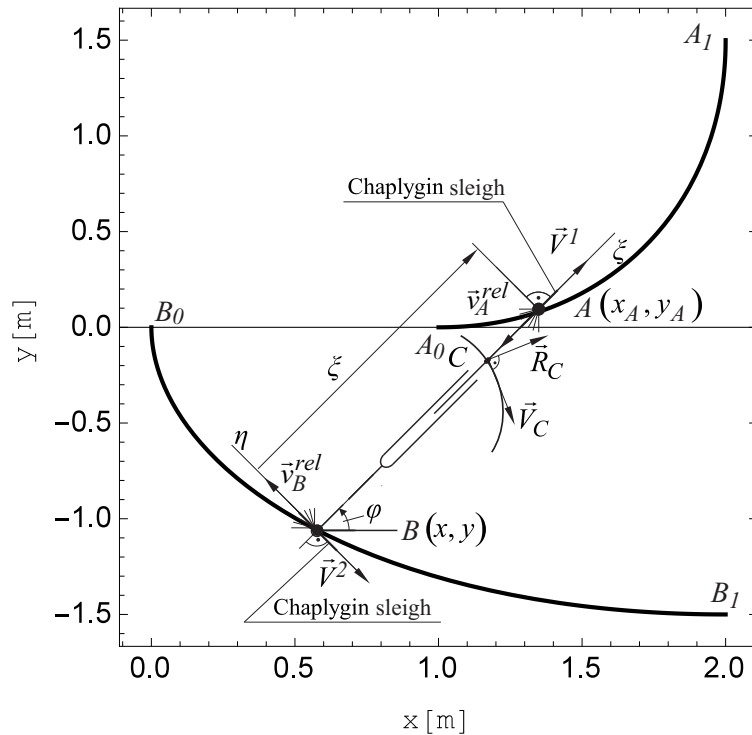


Figure 1: Variable mass nonholonomic mechanical system and the law of the points A and B motion.

The configuration of the considered system is defined by a set of Lagrangian coordinates $q = (q^1, q^2, q^3, q^4)$, where $q^1 \triangleq x$ and $q^2 \triangleq y$ are Cartesian coordinates of the material point B , $q^3 \triangleq \varphi$ is the angle between the axes Ox and $B\xi$ and $q^4 \triangleq \xi$ is the relative coordinate of the material point A relative to the non-stationary coordinate system.

Changes in masses of the material points A and B are specified in the following form:

$$\begin{aligned} m_A(t) &= m_0 e^{-k_A t}, \\ m_B(t) &= m_0 e^{-k_B t}, \end{aligned} \tag{64}$$

where m_0 is mass of the material points A and B at the initial instant of time, and k_A and k_B are defined positive constants. Without loss of generality, the magnitudes of relative velocities of the particles' expelling from the material points A and B are constant and mutually equal:

$$v_A^{rel} = v_B^{rel} = v_r, \tag{65}$$

where v_r is a defined positive constant, and $\vec{v}_A^{rel} = -v_r \vec{\lambda}$ and $\vec{v}_B^{rel} = v_r \vec{\mu}$. According to the restriction of motion of the material points A and B , and in accordance with (4), nonholonomic homogeneous constraints

can be written in the following manner:

$$\begin{aligned}\gamma^3 &\equiv \dot{q}^1 \cos(q^3) + \dot{q}^2 \sin(q^3), \\ \gamma^4 &\equiv -\dot{q}^1 \sin(q^3) + \dot{q}^2 \cos(q^3) + q^4 \dot{q}^3.\end{aligned}\quad (66)$$

For independent quasi-velocities, the velocities of the material points A and B are taken:

$$\begin{aligned}V^1 &= V_A = \dot{q}^4, \\ V^2 &= V_B = \dot{q}^1 \sin(q^3) - \dot{q}^2 \cos(q^3),\end{aligned}\quad (67)$$

Now, according to (7), (66) and (67), all generalized velocities can be expressed via independent quasi-velocities:

$$\begin{aligned}\dot{q}^1 &= \sin(q^3)V^2, \\ \dot{q}^2 &= -\cos(q^3)V^2, \\ \dot{q}^3 &= \frac{1}{q^4}V^2, \\ \dot{q}^4 &= V^1.\end{aligned}\quad (68)$$

The kinetic energy of the system, according to (8), is written in the following form:

$$T^* = \frac{1}{2}(m_A V_A^2 + m_B V_B^2).\quad (69)$$

At point C of the system, an ideal holonomic stationary constraint is imposed in the form of smooth guides, so control was accomplished without active control forces by means of the constraint reaction \vec{R}_C , realizing the constraint in such way that the condition $\vec{R}_C \cdot \vec{v}_C = 0$ is satisfied during brachistochronic motion. Accordingly, the line of the guide path coincides with the line of the material point C path, positioned in the AB direction, and therefore the parametric equations of the guide line are specified in the form as follows:

$$\begin{aligned}x_C(t) &= q^1 + (q^4 - \overline{AC} \cos(q^3)), \\ y_C(t) &= q^2 + (q^4 - \overline{AC} \sin(q^3)).\end{aligned}\quad (70)$$

Now, based on (15)-(23), (27), (31), (32) and (64)-(70), differential equations of motion of the system can be constructed:

$$\begin{aligned}\dot{V}^1 &= c^1 + d^1 u = k_A v_r + \frac{\overline{AC} V^2 u}{m_A \sqrt{(\overline{AC} V^2)^2 + (q^4 V^1)^2}}, \\ \dot{V}^2 &= c^2 + d^2 u = k_B v_r - \frac{\overline{AC} V^1 u}{m_B \sqrt{(\overline{AC} V^2)^2 + (q^4 V^1)^2}}.\end{aligned}\quad (71)$$

Afterwards, a rheonomic coordinate can be introduced and (36) and (37) can be defined applying (68) and (71), as well as all other needed quantities so as to solve the formulated problem.

For initial and end conditions (24), (25) and (26) it is taken:

$$\begin{aligned}t_0 &= 0, \quad q^1(t_0) = 0, \quad q^2(t_0) = 0, \quad q^3(t_0) = 0, \quad q^4(t_0) = a, \\ T^*(t_0) + \Pi(t_0) &= \frac{1}{2}(m_A(t_0)V_A^2(t_0) + m_B(t_0)V_B^2(t_0)) = E_0, \\ q^1(t_f) &= 2a, \quad q^2(t_f) = -1.5a, \quad q^3(t_f) = \pi/2, \quad q^4(t_f) = 3a.\end{aligned}\quad (72)$$

Using the numerical procedure described in the preceding Section, the solution of the problem was found for the following parameters:

$$E_0 = 100 \frac{\text{kgm}^2}{\text{s}^2}, a = 1\text{m}, k_A = 0.5 \frac{1}{\text{s}}, k_B = 0.25 \frac{1}{\text{s}}, v_r = 20 \frac{\text{m}}{\text{s}}, m_0 = 100\text{kg}, \overline{AC} = 1/3\text{m}.\quad (73)$$

Since the constraint without restricted reaction cannot be realized in reality, the example examines change in the scalar control structure with decrease in the range of restrictions in the constraint reaction. The numerical procedure gives solutions for the system of differential equations of motion, as well as for the costate system in numerical form:

$$q^1(t), q^2(t), q^3(t), q^4(t), V^1(t), V^2(t), \lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), v_1(t), v_2(t), \tag{74}$$

and the time of brachistochronic motion t_f , and corresponding times t_1 and t_2 which correspond to disruptions, where junction occurs between singular and nonsingular controls. Figure 1 shows trajectories of the material points A and B .

N°	$ R_C^* [\text{N}]$	$t_f[\text{s}]$	$V_f^1[\text{m/s}]$	$V_f^2[\text{m/s}]$	$\lambda_2[\text{s/m}]$	$\lambda_{3f}[\text{s}]$	$t_1[\text{s}]$	$t_2[\text{s}]$
1	> 1657.38	0.70928	5.677637	6.074301	0.019224	0.096375	/	/
2	1630	0.70928	5.677809	6.074164	0.019217	0.096401	0.132072	0.017985
3	1604.96	0.709281	5.678575	6.073552	0.019184	0.096521	0.160024	/
4	1400	0.709375	5.736696	6.027092	0.01334	0.108238	0.314127	/
5	1327	0.709494	5.806047	5.97074	0.004958	0.123753	0.378098	/
6	1300	0.70956	5.84072	5.942111	-0.000717	0.133814	0.664786	0.407962
7	1242.55	0.709782	5.905594	5.887498	-0.05321	0.223104	0.524526	/

Table 1: Numerical solutions of TPBVP when the restriction of the constraint reaction changes.

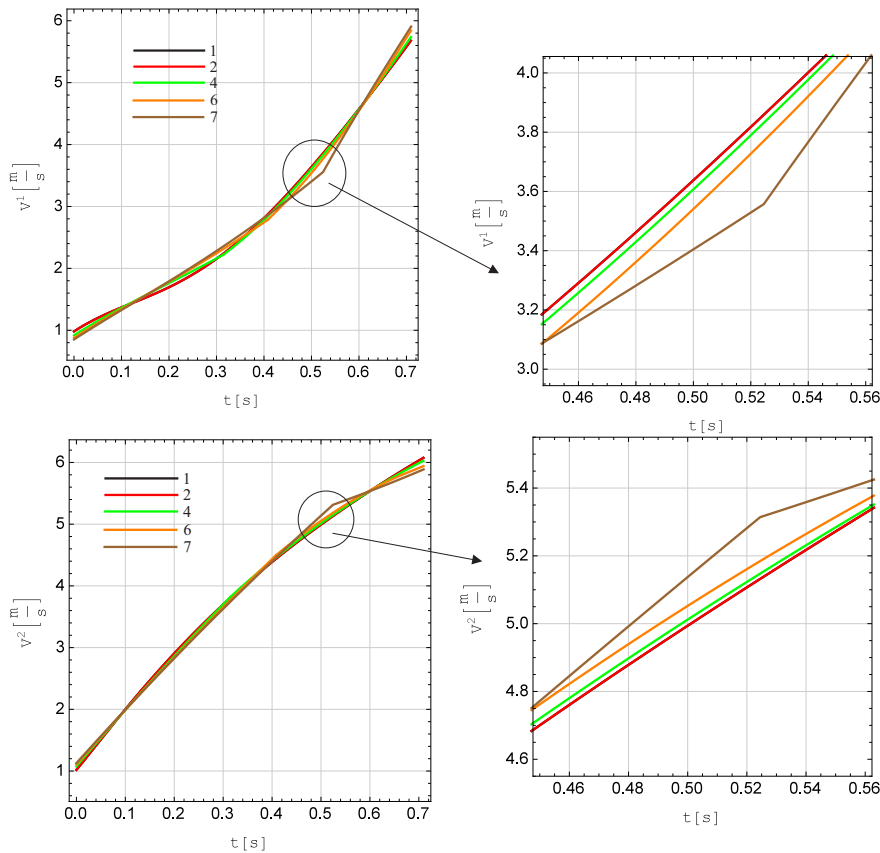


Figure 2: Graphs of velocities V^1 and V^2 .

Table 1 displays values of the missing boundary values for different restriction values (40). It is evident that as the range of restrictions decreases (40), the time of brachistochronic motion increases t_f , the material point A velocity increases and the material point B velocity decreases. Values 1 correspond to the restriction in which singular control is over the entire interval of motion. Values 2 correspond to the restriction in which the structure of the singular – minimal – singular control type occurs. Such control holds until the boundary is reduced to values 3, which represent a boundary value in which one singular value disappears, and therefore the structure of control is minimal – singular, and values 4 also have such type of control. Values 5 correspond to the restriction of the constraint reaction, representing a boundary value in which the restriction also occurs from the upper boundary, and hence the minimal- singular – maximal control structure starts. This type of control also occurs with further decrease of the boundary shown by values 6. Values 7 correspond to the restriction that matches a new boundary in which the second part of singular control disappears too, so that we have the minimal – maximal control structure that also occurs with lower restriction values of the constraint reaction. Further decrease of the boundary by the specific value leads to the occurrence of the structure of the bang-bang type of control.

Figure 2 shows a comparative graphic representation of the values of velocities V^1 and V^2 for several different values of the constraint reaction corresponding to the numbers from Table 1. Figure 3 displays graphs of control for corresponding values from Table 1.

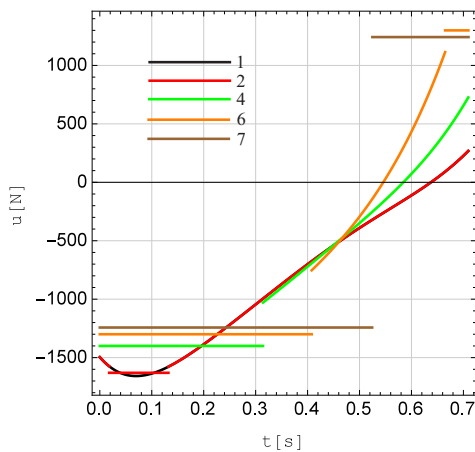


Figure 3: Graph of control $u(t)$.

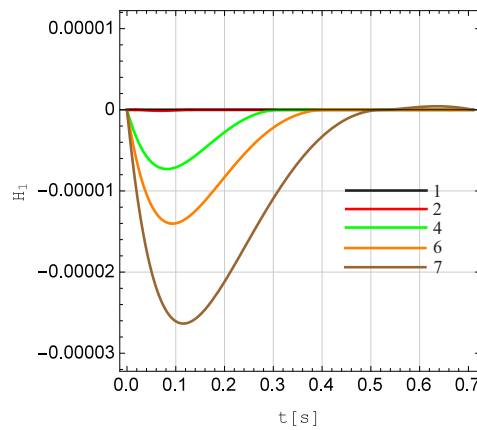


Figure 4: Graph of discontinuous function H_1 .

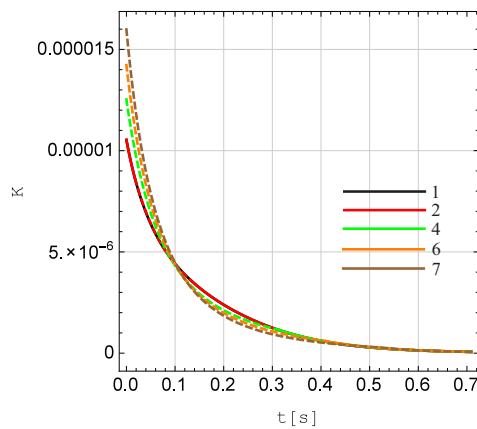


Figure 5: Evidence of Kelley's optimality condition.

Figure 4 gives a comparative representation of the function H_1 , the so-called switching function, for several restriction values of the constraint reaction. Thus, fulfillment of the expression (41) is evident.

Since control in this example is the first order singular control, it is needed to satisfy Kelley's optimality condition (61). Figure 5 presents the law of change in the function K for different restriction values of the constraint, which indicates the fulfillment of Kelley's optimality condition.

5. Conclusion

The present work has solved the problem of realizing brachistochronic planar motion of a nonholonomic variable mass mechanical system by means of an ideal holonomic constraint with restricted reaction. Considerations presented in this work rely on the work [8] and thus are a kind of continuation of mentioned study. The considered system has two degrees of freedom so that the motion can be realized by means of a single ideal holonomic constraint. The restricted reaction of the ideal holonomic constraint has been deployed as the control, and hence the brachistochrone problem is formulated as an optimal control task. Due to the restriction of the constraint reaction, the work examined how decrease in the range of permissible control affects change in the control structure, and hence transition from the singular control to a combination of the singular-nonsingular control, as long as the bang-bang type of control occurs.

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