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Error Estimations of Turán Formulas with Gori-Micchelli and Generalized Chebyshev Weight Functions

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Abstract. S. Li in [Studia Sci. Math. Hungar. 29 (1994) 71–83] proposed a Kronrod type extension to the well-known Turán formula. He showed that such an extension exists for any weight function. For the classical Chebyshev weight function of the first kind, Li found the Kronrod extension of Turán formula that has all its nodes real and belonging to the interval of integration, $[-1,1]$. In this paper we show the existence and the uniqueness of the additional two cases – the Kronrod exstensions of corresponding Gauss-Turán quadrature formulas for special case of Gori-Micchelli weight function and for generalized Chebyshev weight function of the second kind, that have all their nodes real and belonging to the integration interval $[-1,1]$. Numerical results for the weight coefficients in these cases are presented, while the analytic formulas of the nodes are known.

1. Introduction

Let $d\lambda$ be given nonnegative measure on the real line R with compact or unbounded support, for which all moments $\mu_k = \int_R t^k \, d\lambda(t)$ for $k \in N_0$ exist and are finite, and $\mu_0 > 0$. If λ is an absolutely continuous function, then $d\lambda(t) = \omega(t)\, dt$, where ω is a given nonnegative and integrable weight function over the smallest interval $[a,b]$ which contains the support of λ. The results of this paper concern with the case $d\lambda(t) = \omega(t)\, dt$, for certain weight functions $\omega(t)$. As usual, let P_k denote the set of polynomials of degree at most k.

In 1950 P. Turán [23] proposed an interpolatory quadrature formula with multiple nodes which has the highest possible algebraic degree of precision (ADP),

$$\int_a^b f(t)\omega(t)\, dt \approx \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) \quad (s \in N_0). \tag{1}$$

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This quadrature formula has been named the Gauss-Turán quadrature formula. Turán himself considered the case when $\omega(t) = 1$ and $[a,b] = [-1,1]$.

The polynomial $\pi_n(t)$ of degree n, known as s-orthogonal polynomial, which satisfies the orthogonality relation

$$\int_a^b \pi_n^{2s+1}(t)t^k\omega(t)\,dt = 0, \quad \text{for all } k = 0,\dots,n-1,$$

is connected with the Gauss-Turán quadrature formula (1). It is well known that the zeros of a polynomial π_n are the nodes τ_ν of the generalized Gauss-Turán quadrature formula (1). The weight coefficients $A_{i,\nu}$ from (1) are determined through interpolation and they are not all positive in general. *ADP* of the formula (1) is $2(s+1)n-1$. Numerically stable methods for constructing nodes τ_ν and coefficients $A_{i,\nu}$ in Gauss-Turán quadrature formulas with multiple nodes (1), and their generalizations, are treated in several papers; see [3], [4], [12], [13], [15], [22]; see also the book [2]. For more details on Gauss-Turán quadrature formulas and corresponding s-orthogonal polynomials, including answers on the questions of their existence and uniqueness, see [5], [14], [6], [18], [19].

In this paper we consider an optimal extension of generalized Gauss-Turán formula in the sense of Kronrod (see [8] for optimal extension of the ordinary Gaussian quadrature).

The estimation of the error in a quadrature formula is an important problem. The purpose of this paper is to consider a Kronrod extension for Gauss-Turán quadrature formulas. These extensions are applied to estimate the error in the original Gauss-Turán quadrature. The constructions of these extensions for Gauss-Turán quadrature formulas with respect to one Gori-Micchelli weight function and generalized Chebyshev weight function of the second kind are proposed. That way, we extend the results by Li from the paper [11] and give numerical examples showing that the proposed method provides an efficient error estimations.

2. Kronrod extension of Gauss-Turán quadrature formula

Following Kronrod's idea, Li [11] considered an extension of the formula (1) to the following formula

$$\int_a^b f(t)\omega(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{j=1}^{n+1} C_j f(\hat{\tau}_j) \quad (s \in N_0), \tag{2}$$

where τ_ν are the same nodes as in (1), and the new nodes $\hat{\tau}_j$ and new weights $B_{i,\nu}, C_j$ are chosen to maximize *ADP* of (2). It is shown in the paper [11], when ω is any weight function on $[a,b]$, that we can always obtain the maximum degree $2n(s+1)+n+1$ of the quadrature formula (2) by taking $\hat{\tau}_j$ to be the zeros of the polynomials $\hat{\pi}_{n+1}$ satisfying the orthogonality property

$$\int_a^b \hat{\pi}_{n+1}(t)\pi_n^{2s+1}(t)p(t)\omega(t)\,dt = 0, \quad \text{for all } p \in P_n.$$

It is shown in [11] that the so called Stieltjes polynomial of degree $n+1$, $\hat{\pi}_{n+1}$ always exists and is unique up to a multiplicative constant.

It is well known (see [5]) that the nodes in Gauss-Turán quadrature formula are all real, distinct and internal in the interval $[a,b]$.

We need a couple of facts concerning the theory of quadratures with multiple nodes which contain Gauss-Turán quadrature formulas. We recall the following theorem established by Ghizzetti and Ossicini [6].

Theorem 2.1. *For any given set of odd multiplicities $\nu_1, \dots, \nu_n$, i. e. $\nu_j = 2s_j + 1$, $s_j \in N_0$, $j = 1, \dots, n$, there exists a unique quadrature formula of the form*

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \quad a \le x_1 < \cdots < x_n \le b, \tag{3}$$

of ADP $= \nu_1 + \cdots + \nu_n + n - 1$, which is the well known Chakalov-Popoviciu quadrature formula. The nodes $x_1, \dots, x_n$ of this quadrature are determined uniquely by the orthogonality property

$$\int_a^b \omega(t) \prod_{k=1}^{n} (t - x_k)^{\nu_k} p(t)\, dt = 0, \quad \text{for all } p \in P_{n-1}.$$

The corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is known in the classical literature as σ-orthogonal polynomial, with $\sigma = (s_1, \dots, s_n)$, where n indicates the size of the array, and $\nu_k = 2s_k + 1$, $s_k \in N_0$, $k = 1, \dots, n$, in the preceding formula.

Bojanov and Petrova [1] stated and proved the following important theorem which reveals the relation between the standard interpolatory quadratures of the type (3) and the quadratures for Fourier coefficients.

Theorem 2.2. *For any given sets of multiplicities $\bar{\mu} := (\mu_1, \dots, \mu_k)$ and $\bar{\nu} := (\nu_1, \dots, \nu_n)$, and nodes $y_1 < \cdots < y_k$, $x_1 < \cdots < x_n$, there exists a quadrature formula of the form*

$$\int_a^b \omega(t) \Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \tag{4}$$

where $\Lambda^{\bar{\mu}}(t; \mathbf{y}) := \prod_{m=1}^{k} (t - y_m)^{\mu_m}$, with ADP $= N$ if and only if there exists a quadrature formula of the form

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{m=1}^{k} \sum_{\lambda=0}^{\mu_m - 1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \tag{5}$$

which has ADP $= N + \mu_1 + \cdots + \mu_k$. In the case $y_m = x_j$ for some m and j, the corresponding terms in both sums combine in one sum of the form

$$\sum_{\lambda=0}^{\mu_m + \nu_j - 1} d_{m\lambda} f^{(\lambda)}(y_m).$$

3. Gori-Micchelli weight

In [16] it is considered a subclass of the Gori-Micchelli weight functions (cf. [7]),

$$\omega_{n,\ell}(t) = \left[\frac{U_{n-1}(t)}{n} \right]^{2\ell} (1 - t^2)^{\ell - 1/2}, \quad \ell \in \{0, 1, \dots, s\}, \quad s \in N; \quad t \in [-1, 1]. \tag{6}$$

In the particular case $\ell = 0$, (6) reduces to the Chebyshev weight function of first kind $\omega_{n,0}(t) = (1 - t^2)^{-1/2}$.

Recall that for the weight functions (6), the Chebyshev polynomials of first kind T_n are s-orthogonal (cf. [7]). Milovanović and Spalević in [16] for the quadrature formula with multiple nodes

$$\begin{aligned} \int_{-1}^{1} f(t)\, \omega_{n,\ell}(t)\, dt \quad &\approx \quad \sum_{\nu=1}^{n} \sum_{i=0}^{2s} c_{\nu i} f^{(i)}(x_\nu) + \sum_{\mu=2}^{n} \sum_{j=0}^{2(s-\ell)} c^*_{\mu j} f^{(j)}(x^*_\mu) \\ &\quad + \sum_{j=0}^{s-\ell} \left(c^*_{1,j} f^{(j)}(-1) + c^*_{n+1,j} f^{(j)}(1) \right), \end{aligned} \tag{7}$$

proved the following statement.

Theorem 3.1. *In the Kronrod extension* (7) *of the Gauss-Turán quadrature formula of type* (1),

$$\int_{-1}^{1} f(t)\,\omega_{n,\ell}(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s} \bar{c}_{\nu i} f^{(i)}(x_\nu), \tag{8}$$

where $s_\nu = s$, $\nu = 1,\dots,n$, with the weight function (6), *and for $n \geq 2$, the corresponding generalized Stieltjes polynomial $E_{n+1}^{(\sigma^*)}(t)$ is given by $E_{n+1}^{(\sigma^*)}(t) \equiv (t^2-1)\,U_{n-1}(t)$, i.e., the nodes x_μ^*, $\mu = 2,\dots,n$, are the zeros of the Chebyshev polynomial of second kind $U_{n-1}(t)$ and $x_1^* = -1, x_{n+1}^* = 1$.*

The zeros of $T_n(t)$ and $E_{n+1}^{(\sigma^*)}(t)$ interlace (i.e., satisfy the well known interlacing property, since $2(t^2-1)U_{n-1}(t) = 2n^{-1}(t^2-1)T_n'(t)$ (cf. [20, Lemma 1, p. 180]).

The algebraic degree of precision of the quadrature formula (7) as a Kronrod extension is $N = n(4s - 2\ell + 3) + 1$.

In the special case when $\ell = s$, the Kronrod extension (7) reduces to the formula of type (2), i. e.

$$\begin{aligned}\int_{-1}^{1} f(t)\,\omega_{n,s}(t)\,dt \quad \approx \quad & \sum_{\nu=1}^{n}\sum_{i=0}^{2s} c_{\nu i} f^{(i)}(x_\nu) + \sum_{\mu=2}^{n} c_{\mu 0}^* f(x_\mu^*) \\ & + c_{1,0}^* f(-1) + c_{n+1,0}^* f(1),\end{aligned} \tag{9}$$

which has the algebraic degree of precision is $N = n(2s+3)+1$, where

$$\omega_{n,s}(t) = \left[\frac{U_{n-1}(t)}{n}\right]^{2s} (1-t^2)^{s-1/2}, \quad s \in N.$$

In this way we have proved the following statement.

Theorem 3.2. *Let be given $n \in N \setminus \{1\}, s \in N$. In the Kronrod extension* (9) *of type (2) of the Gauss-Turán quadrature formula (8) with the weight function $\omega_{n,s}$, i. e.*

$$\int_{-1}^{1} f(t)\,\omega_{n,s}(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s} \bar{c}_{\nu i} f^{(i)}(x_\nu), \tag{10}$$

the corresponding generalized Stieltjes polynomial $E_{n+1}^{(\sigma^*)}(t)$ is given by $E_{n+1}^{(\sigma^*)}(t) \equiv (t^2-1)\,U_{n-1}(t)$, i.e., the nodes x_μ^*, $\mu = 2,\dots,n$, are the zeros of the Chebyshev polynomial of second kind $U_{n-1}(t)$ and $x_1^* = -1, x_{n+1}^* = 1$.

By applying Theorem 2.2 and Theorem 3.2 we deduce as follows.

Corollary 3.3. *Let be given $n \in N \setminus \{1\}, s \in N$. In the Kronrod extension*

$$\begin{aligned}\int_{-1}^{1} f(t)T_n(t)\,\omega_{n,s}(t)\,dt \quad \approx \quad & \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} \widehat{c}_{\nu i} f^{(i)}(x_\nu) + \sum_{\mu=2}^{n} \widehat{c}_{\mu 0}^* f(x_\mu^*) \\ & + \widehat{c}_{1,0}^* f(-1) + \widehat{c}_{n+1,0}^* f(1),\end{aligned} \tag{11}$$

of the Gaussian quadrature formula

$$\int_{-1}^{1} f(t)T_n(t)\,\omega_{n,s}(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} \widetilde{c}_{\nu i} f^{(i)}(x_\nu), \tag{12}$$

the corresponding generalized Stieltjes polynomial $E_{n+1}^{(\sigma^*)}(t)$ is given by $E_{n+1}^{(\sigma^*)}(t) \equiv (t^2-1)\,U_{n-1}(t)$, i. e., the nodes x_μ^*, $\mu = 2,\dots,n$, are the zeros of the Chebyshev polynomial of second kind $U_{n-1}(t)$ and $x_1^* = -1, x_{n+1}^* = 1$.

It is clear that $ADP(12) = (2s+1)n - 1$ and $ADP(11) = 2n(s+1) + 1$.

Table 1: Coefficients $c_{\nu,i}$ of the formula (8) in the case $n = s = 2$

(ν, i)	$c_{\nu,i}$	(ν, i)	$c_{\nu,i}$
(1,0)	3.68155389092553895l323 (-02)	(2,0)	3.68155389092553895l3237 (-02)
(1,1)	6.05900058895727513621 5 (-04)	(2,1)	- 6.05900058895727513621 5(-04)
(1,2)	4.28436040366497452831 4 (-04)	(2,2)	4.28436040366497452831 48 (-04)
(1,3)	1.27111900467635142717 8 (-05)	(2,3)	- 1.27111900467635142717 8 (-05)
(1,4)	1.49802811316957151339 6 (-06)	(2,4)	1.49802811316957151339 61 (-06)

3.1. *Numerical construction of the coefficients in the Kronrod extension of Gauss-Turán quadrature formula for Gori-Micchelli weight function*

Theorem 2.5 from [13] states that for fixed ν, $1 \le \nu \le n$, the coefficients $c_{\nu,i}$ in generalized Gauss-Turán quadrature formula (1) can be determined by solving the system

$$\begin{aligned} b_{2s_\nu+1} &= (2s_\nu)! c_{\nu,2s_\nu} = \hat{\mu}_{\nu,2s_\nu}, \\ b_k &= (k-1)! c_{\nu,k-1} = \hat{\mu}_{\nu,k-1} - \sum_{j=k+1}^{2s_\nu+1} \hat{a}_{k,j} b_j, \quad k = 2s_\nu, \ldots, 1, \end{aligned}$$

where

$$\hat{\mu}_{\nu,k} = \int_R (t - x_\nu)^k \prod_{i \ne \nu} \Big(\frac{t - x_i}{x_\nu - x_i}\Big)^{2s_\nu+1} \omega(t)\, dt, \tag{13}$$

$$\hat{a}_{k,k+j} = -\frac{1}{j} \sum_{l=1}^{j} u_l \hat{a}_{l,j}, \quad \hat{a}_{k,k} = 1,$$

$$u_l = \sum_{i \ne \nu} (2s_i + 1)(x_i - x_\nu)^{-l}, \quad l = 2s_\nu, \ldots, 1,$$

$\sigma^* = (s_1, \ldots, s_n)$ and s_ν is the multiplicity of the ν-th node, x_ν.

The integrals $\hat{\mu}_{\nu,k}$, $k = 0, \ldots, 2s_\nu$, given by the formula (13) can be computed by applying Gaussian quadrature formula `sgauss(dig,N,ab)` (see [2]). Input argument of the previous function is the matrix `ab` which consists the coefficients of three-term recurrence relation of the corresponding orthogonal polynomials.

In the case of Gori-Micchelli weight function and $l = s_\nu$, the integral $\hat{\mu}_{\nu,k}$ is given by the term

$$\hat{\mu}_{\nu,k} = \int_R (t - x_\nu)^k \prod_{i \ne \nu} \Big(\frac{t - x_i}{x_\nu - x_i}\Big)^{2s_i+1} \left[\frac{U_{n-1}(t)}{n}\right]^{2s_\nu} (1 - t^2)^{s_\nu - 1/2} dt.$$

The coefficients of Gauss-Turán formula

Example 1. We consider the integral of the function $f(t) = e^t$, with respect to the Gori-Micchelli weight function, $n = 2$ and $s = s_\nu = 2$, $\nu = 1, \ldots, n$. The appropriate coefficients are given in Table 1.

By multiplying them with appropriate derivatives of the function at appropriate nodes, we get the approximation of the integral

$$I_{G-T}(f) = 0.09295308146342168336548805217023481677297473284729\ldots$$

Table 2: Coefficients of the formula (9) in the case $n = s = 2$

$c^*_{1,0}$	5.7524279545711546114428284 (-04)	$c^*_{3,0}$	5.7524279545711546114428284 (-04)
$c^*_{2,0}$	1.1504855909142309222885656 (-03)		
(ν, i)	$c_{\nu,i}$	(ν, i)	$c_{\nu,i}$
(1,0)	3.566505331834115859094555 (-02)	(2,0)	3.5665053318341158590945557 (-02)
(1,1)	4.7243256340471061376788 2 (-04)	(2,1)	- 4.7243256340471061376788 2 (-04)
(1,2)	3.3406026923681444748743 5 (-04)	(2,2)	3.3406026923681444748743 50 (-04)
(1,3)	6.3555950233817571358907 9 (-06)	(2,3)	-6.3555950233817571358907 9 (-06)
(1,4)	7.4901405658478575669828 4 (-07)	(2,4)	7.4901405658478575669828 41 (-07)

The coefficients of Kronrod's extension of Gauss-Turán quadrature formula

Beside the nodes x_ν, the zeros of the Chebyshev orthogonal polynomial of the first kind, the formula (9) contains additional nodes - the zeros of Chebyshev polynomial of the second kind, $x^*_\mu = -\cos(j-1)\pi/n$, $\mu = 2, \ldots, n$ and the nodes -1 and 1 which also can be treated as zeros of Chebyshev polynomial of the second kind, $x^*_\mu = -\cos(j-1)\pi/n$ for $\mu = 1$ and $\mu = n+1$, i. e.

$$I_{K,G-T}(f) = \int_{-1}^{1} f(t)\, \omega_{n,s}(t)\, dt \quad \approx \quad \sum_{\nu=1}^{n} \sum_{i=0}^{2s} c_{\nu i} f^{(i)}(x_\nu) + \sum_{\mu=1}^{n+1} c^*_{\mu 0} f(x^*_\mu).$$

The previous formula is formula of Chakalov-Popoviciu type (see [13]), because the nodes x_ν are with multiplicity $2s+1$, and other nodes, x^*_μ, $\mu = 1, 2, \ldots, n+1$ are with multiplicity 1. The nodes x_ν and x^*_μ satisfy *"interlacing property"*, so we can arrange them in increasing order. At the nodes x_ν we consider the values of the function and the values of the derivatives of the function ending with $2s$-th. At the nodes x^*_μ we consider just the values of the function.

Example 2. By applying Kronrod's extension of Gauss-Turán formula (9) with respect to the Gori-Micchelli weight function in the case $f(t) = e^t$, $n = 2$, $\sigma^* = (0, 2, 0, 2, 0)$, $s = s_{2\nu} = 2, \nu = 1, \ldots, n$ we get the following approximation

$$I_{K,G-T}(f) = 0.0929530814634981968283020556958425204614780789630 79\ldots$$

The coefficients of the previous formula are given in Table 2.

The error estimations and the actual errors

In the first example, the integral of the function is approximated by using Gauss-Turán formula, and in the second example, we used Kronrod's extension of mentioned formula. We compared them by applying the well-known method of estimation of the error in the corresponding Gauss-Turán quadrature formula. In the case $n = s = 2$ we get

$$R(f) = |I_{G-T}(f) - I_{K,G-T}(f)| = 7.650824418448110\ldots(-14).$$

For more details concerning the given method of estimation of the error in quadrature formulas see Notaris [17]. Table 3 presents the error estimations $R(f_1)$ and the actual errors *"Error(f_1)"* in the case of the same function, $f_1(t) = e^t$, but for different choices of number of the nodes and their multiplicities. Tables 4 and 5 present the error estimations and the actual errors in the case of functions $f_2(t) = e^{5t}$ and $f_3(t) = e^{10t}$, respectively.

The actual error presents the difference between the Gaussian approximation and Gauss-Turan approximation for fixed n and s (because the weight function depends on them),

$$Error(f) = |I_G(f) - I_{G-T}(f)|.$$

Table 3: Error estimations, $R(f_1)$, and actual errors, $Error(f_1)$, in the case of the function $f_1(t) = e^t$

	$s=1$	$s=1$	$s=2$	$s=2$	$s=3$	$s=3$
n	$R(f_1)$	$Error(f_1)$	$R(f_1)$	$Error(f_1)$	$R(f_1)$	$Error(f_1)$
2	7.823(-08)	7.823(-08)	7.651(-14)	7.651(-14)	2.271(-20)	2.271(-20)
3	1.814(-13)	1.814(-13)	1.756(-22)	1.756(-22)	2.613(-34)	2.613(-34)
4	1.453(-19)	1.453(-19)	8.931(-34)	8.931(-34)	4.274(-49)	4.274(-49)
5	4.985(-26)	4.985(-26)	1.334(-44)	1.334(-44)	1.409(-64)	1.409(-64)
6	8.468(-33)	8.468(-33)	7.160(-56)	7.160(-56)	1.211(-80)	1.211(-80)

Table 4: Error estimations, $R(f_2)$, and actual errors, $Error(f_2)$, in the case of the function $f_2(t) = e^{5t}$

	$s=1$	$s=1$	$s=2$	$s=2$	$s=3$	$s=3$
n	$R(f_2)$	$Error(f_2)$	$R(f_2)$	$Error(f_2)$	$R(f_2)$	$Error(f_2)$
2	5.790(-02)	5.779(-02)	2.931(-05)	2.927(-05)	4.901(-09)	4.901(-09)
3	6.696(-05)	6.696(-05)	9.162(-11)	9.162(-11)	2.082(-17)	2.082(-17)
4	3.144(-08)	3.144(-08)	6.939(-17)	6.939(-17)	1.193(-26)	1.193(-26)
5	6.314(-12)	6.314(-12)	1.507(-23)	1.507(-23)	1.483(-36)	1.483(-36)
6	6.409(-16)	6.409(-16)	1.225(-30)	1.225(-30)	4.860(-47)	4.860(-47)

Table 5: Error estimations, $R(f_3)$, and actual errors, $Error(f_3)$, in the case of the function $f_3(t) = e^{10t}$

	$s=1$	$s=1$	$s=2$	$s=2$	$s=3$	$s=3$
n	$R(f_3)$	$Error(f_3)$	$R(f_3)$	$Error(f_3)$	$R(f_3)$	$Error(f_3)$
2	8.636(01)	8.277(01)	4.409(-01)	4.336(-01)	8.991(-04)	8.925(-04)
3	1.085(00)	1.084(00)	6.204(-05)	6.203(-05)	6.904(-10)	6.904(-10)
4	5.900(-03)	5.900(-03)	2.360(-09)	2.360(-09)	1.110(-16)	1.110(-16)
5	1.572(-05)	1.572(-05)	2.935(-14)	2.935(-14)	2.566(-24)	2.566(-24)
6	2.237(-08)	2.237(-08)	1.389(-19)	1.389(-19)	2.000(-32)	2.000(-32)

4. Generalized Chebyshev weight of the second kind

In the paper [10], for every $s > 0$, Micchelli and Sharma constructed a quadrature formula of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t) f(t)\, dt \approx \sum_{j=0}^{s} \left[A_j f^{(j)}(-1) + B_j f^{(j)}(1) \right] + \sum_{j=1}^{n-1} a_{ji} f^{(i)}(x_j), \tag{14}$$

with $ADP(14) = (2s+3)n - 1$, which has the highest possible precision. The nodes of their formula are located at the extremal points $-1, x_1, \ldots, x_{n-1}, 1$ of the Chebyshev polynomial of the first kind T_n.

By proving the uniqueness of the formula (14), Bojanov and Petrova in the paper [1] got the quadrature formula

$$I^*_{K,G-T}(f) = \int_{-1}^{1} (1-t^2)^{1/2+s} f(t)\, dt \approx \sum_{\nu=1}^{n-1} \sum_{i=0}^{2s} c_{\nu,i} f^{(i)}(x^*_\nu) + \sum_{\mu=1}^{n} c^*_{\mu,0} f(x_\mu) \tag{15}$$

with respect to the generalized Chebyshev weight function of second kind $\omega_2(t) = (1-t^2)^{1/2+s}$ on $[-1, 1]$ (cf. [19]), which integrate exactly all polynomials of degree $(2s+4)n - 1 - (2s+2) = (2s+2)(n-1) + 2n - 1$. This formula is based on n interior simple nodes $x_1, \ldots, x_n$, the zeros of the Chebyshev polynomial of the first kind T_n, and $(n-1)$ nodes $x^*_1, \ldots, x^*_{n-1}$, the zeros of the Chebyshev polynomial of second kind U_{n-1}, each with odd multiplicity $2s+1$. Such formula can attain the highest possible ADP, equal to $ADP = 2n + (2s+2)(n-1) - 1$, only in the case when all nodes are Gaussian.

Quadrature formula (15) is in fact Kronrod extension of type (2) of the Gauss-Turán quadrature formula

$$I^*_{G-T}(f) = \int_{-1}^{1} (1-t^2)^{1/2+s} f(t)\, dt \approx \sum_{\nu=1}^{n-1} \sum_{i=0}^{2s} \bar{c}_{\nu,i} f^{(i)}(x^*_\nu), \tag{16}$$

with $ADP(16) = 2(n-1)(s+1) - 1$.

Hence, we deduce as follows.

Theorem 4.1. *Let be given $n \geq 2, s \in N$. In the Kronrod extension* (15) *of the type (2) of the Gauss-Turán quadrature formula (16) with the generalized Chebyshev weight of the second kind, the corresponding generalized Stieltjes polynomial $E_n^{(\sigma^*)}(t)$ is given by $E_n^{(\sigma^*)}(t) \equiv T_n(t)$, i.e., the nodes x_μ, $\mu = 1, \ldots, n$, are the zeros of the Chebyshev polynomial of the first kind, $T_n(t)$.*

4.1. *The coefficients of Kronrod extension of the Gauss-Turán quadrature formula for generalized weight function ω_2*

Again, by applying the Theorem 2.5 from the paper [13] we can compute the coefficients $c_{\nu,i}$ of generalized Gauss-Turán quadrature formula (16) and its Kronrod's extensions, the formula (15).

The integrals $\hat{\mu}_{\nu,k}$, $k = 0, \ldots, 2s_\nu$ given by the formula (13) in the case of generalized Chebyshev weight function of the second kind take a simple form

$$\hat{\mu}_{\nu,k} = \int_R (t - x_\nu)^k \prod_{i \neq \nu} \Big(\frac{t - x_i}{x_\nu - x_i} \Big)^{2s_\nu + 1} (1-t^2)^{s_\nu + 1/2} dt.$$

Example 3. The integrals given by the formulas (15) and (16) are approximated in the cases of the functions $f_1(t) = e^t$, $f_2(t) = e^{5t}$ and $f_3(t) = e^{10t}$. The error estimation is given in the standard way,

$$R^*(f) = |I^*_{G-T}(f) - I^*_{K,G-T}(f)|.$$

The actual error of formula with n nodes, $Error^*$, presents the difference between Gauss-Turán quadrature formula with large number of nodes (e.g. 100) and Gauss-Turán quadrature formula with n nodes,

$$Error^*(f,n) = |I^*_{G-T}(f,100) - I^*_{G-T}(f,n)|.$$

The results are presented in the Tables 6, 7 and 8.

Table 6: Error estimations $R^*(f_1)$ and actual errors, $Error^*(f_1)$, in the case of the function $f_1(t) = e^t$

	$s=1$	$s=1$	$s=2$	$s=2$	$s=3$	$s=3$
n	$R^*(f_1)$	$Error^*(f_1)$	$R^*(f_1)$	$Error^*(f_1)$	$R^*(f_1)$	$Error^*(f_1)$
2	1.161(-07)	1.161(-07)	5.058(-13)	5.058(-13)	6.317(-19)	6.317(-19)
3	6.089(-13)	6.087(-13)	5.904(-22)	5.904(-22)	8.310(-32)	8.310(-32)
4	8.690(-19)	8.690(-19)	9.504(-32)	9.504(-32)	7.645(-46)	7.609(-46)

Table 7: Error estimations $R^*(f_2)$ and actual errors, $Error^*(f_2)$, in the case of the function $f_2(t) = e^{5t}$

	$s=1$	$s=1$	$s=2$	$s=2$	$s=3$	$s=3$
n	$R^*(f_2)$	$Error^*(f_2)$	$R^*(f_2)$	$Error^*(f_2)$	$R^*(f_2)$	$Error^*(f_2)$
2	6.711(-02)	6.711(-02)	1.603(-04)	1.603(-04)	1.171(-07)	1.171(-07)
3	2.059(-04)	2.059(-04)	2.806(-09)	2.806(-09)	5.845(-15)	5.845(-15)
4	1.742(-07)	1.742(-07)	7.105(-15)	7.105(-15)	2.045(-23)	2.045(-23)

Table 8: Error estimations, $R^*(f_3)$, and actual errors, $Error^*(f_3)$, in the case of the function $f_3(t) = e^{10t}$

	$s=1$	$s=1$	$s=2$	$s=2$	$s=3$	$s=3$
n	$R^*(f_3)$	$Error^*(f_3)$	$R^*(f_3)$	$Error^*(f_3)$	$R^*(f_3)$	$Error^*(f_3)$
2	5.473(00)	5.455(00)	1.443(00)	1.443(00)	1.389(-02)	1.389(-02)
3	2.254(00)	2.254(00)	1.433(-03)	1.433(-03)	1.636(-07)	1.636(-07)
4	2.623(-02)	2.623(-02)	2.016(-07)	2.016(-07)	1.351(-13)	1.351(-13)

References

[1] B. Bojanov, G. Petrova, Quadrature formulas for Fourier coefficients, J. Comput. Appl. Math. 231 (2009) 378–391.
[2] W. Gautschi, Orthogonal polynomials: Computation and approximation. New York: Oxford University Press (2004) [Gautschi, W: OPQ suite (http://www.cs.purdue.edu/archives/2001/wxg/codes)]
[3] W. Gautschi, High-precision Gauss-Turán quadrature rules for Laguerre and Hermite weight functions, Numer. Algor. 67 (2014) 59–72.
[4] W. Gautschi, G.V. Milovanović, S-orthogonality and construction of Gauss–Turán-type quadrature formulae, J. Comput. Appl. Math. 86 (1997) 205–218.
[5] A. Ghizzetti, A. Ossicini, Quadrature formulas, Akademie – Verlag, Berlin, 1970.
[6] A. Ghizzetti, A. Ossicini, Sull' esistenza e unicitá delle formule di quadratura gaussiane, Rend. Mat. 8 (1975) 1–15.
[7] L. Gori, C. A. Micchelli, On weight functions which admit explicit Gauss–Turán quadrature formulas, Math. Comp. 65 (1996) 1567–1581.
[8] A. S. Kronrod, Integration with control of accuracy, Soviet Phys. Dokl. 9 (1964) 17–19.
[9] C. A. Micchelli, T. J. Rivlin, Turán formulas and highest precision quadrature rules for Chebyshev coefficients. IBM J. Res. Develop, 16 (1972) 372–379.
[10] C. A. Micchelli, A. Sharma, On a problem of Turán: multiple node Gaussian quadrature, Rend. Mat. 3 (1983) 529–552.
[11] S. Li, Kronrod extension of Turán formula, Studia Sci. Math. Hungarica 29 (1994) 71–83.
[12] G. V. Milovanović, M. M. Spalević, A numerical procedure for coefficients in generalized Gauss-Turán quadratures, FILOMAT 9 (1995) 1–8.
[13] G. V. Milovanović, M. M. Spalević, Construction of Chakalov-Popoviciu's type quadrature formulas, Rend. Circ. Mat. Palermo 52 (1998) 625–636.
[14] G. V. Milovanović, Quadratures with multiple nodes, power orthogonality, and moment-preserving spline approximation. In Numerical analysis 2000, vol. V, Quadrature and orthogonal polynomials, (W. Gautschi, F. Marcellan, L. Reichel, eds.) J. Comput. Appl. Math. 127 (2001) 267–286.
[15] G. V. Milovanović, M. M. Spalević, A.S. Cvetković, Calculation of Gaussian type quadratures with multiple nodes, Math. Comput. Modelling. 39 (2004) 325–347.
[16] G. V. Milovanović, M. M. Spalević, Kronrod extensions with multiple nodes of quadrature formulas for Fourier coefficients, Math. Comp. 83 (2014) 1207–1231.
[17] S. E. Notaris, Gauss-Kronrod quadrature formulae–a survey of fifty years of research, Electron. Trans. Numer. Anal. 45 (2016) 371–404.
[18] A. Ossicini, Costruzione di formule di quadratura di tipo Gaussiano, Ann. Mat. Pura Appl. (4) 72, 213-237 (1966)
[19] A. Ossicini, F. Rosati, Funzioni caratteristiche nelle formule di quadratura gaussiane con nodi multipli, Boll. Un. Mat. Ital. (4) 11 (1975) 224–237.
[20] Y. G. Shi, Generalized Gaussian Kronrod-Turán quadrature formulas, Acta Sci. Math. (Szeged) 62 (1996) 175–185.
[21] Y. G. Shi, General Gaussian quadrature formulas on Chebyshev nodes, Adv. Math. 27 (1998) 227–239.
[22] Y. G. Shi, G. Xu, Construction of σ-orthogonal polynomials and Gaussian quadrature formulas, Adv. Comput. Math. 2 (2007) 79–94.
[23] P. Turán, On the theory of the mechanical quadrature, Acta Sci. Math. (Szeged) 12 (1950) 30–37.
[24] S. Yang, On a quadrature formula of Gori and Micchelli, J. Comput. Appl. Math. 176 (2005) 35-43.