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BERTOLINO-BAKŠA STABILITY AT NONLINEAR VIBRATIONS OF MOTOR VEHICLES

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Dedicated to Professor Aleksandar Bakša, with gratitude

Abstract. Research of vehicle response to road roughness is particularly important when solving problems related to dynamic vehicle stability. In this paper, unevenness of roads is considered as the source of non-linear vibrations of motor vehicles. The vehicle is represented by an equivalent spatial model with seven degrees of freedom. In addition to solving the response by simulating it within a numerical code, quasi-linearization of nonlinear differential equations of motion is carried out. Solutions of quasi-linear differential equations of forced vibrations are determined using the small parameter method and are indispensable for the study of spatial stability of the vehicle. An optimal stabilization for a simplified two-dimensional model was performed. Spatial stability and internal resonance are considered briefly.

1. Introduction

The vehicle is a complex multi-degree of freedom vibration system. Vehicle suspension system plays an important role in vehicle dynamics in terms of riding comfort and handling stability. It performs multiple tasks during vehicle ride: like connecting the vehicle body with the vehicle axes and receiving, absorbing and transferring the forces that are acting between the tyres and the road to the vehicle body. Consequently, suspension protects the vehicle and its occupants from uncomfortable vibrations.

Traditional, passive suspension design is a compromise between two goals-enhancing the vehicle ride comfort or increasing vehicle stability, because, for example, a small amount of suspension damping will provide a more comfortable ride, but it will significantly reduce the vehicle stability. Thus, the current research focuses on electronically controlled suspensions, which are generally divided

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into semi-active suspensions and active suspensions [1]. At the same time, many control methods have been applied to active and semi-active suspension systems: preview control, adaptive control, optimal control, nonlinear control, H-infinity control, neural network control and fuzzy logic control.

Active suspension systems use the force actuators instead of springs and shock absorbers or the actuator that is installed in parallel with a conventional suspension spring [2]. The accelerations of the sprung and unsprung masses and the operating conditions of the vehicle are monitored by sensors. Analogue signals from the sensors are sent to controller, which is designed to use the prescribed control strategy and to control the operation of the actuators. The force in the actuator is modulated in order to achieve improvements in vehicle performance, ride and handling.

In this paper, road roughness is observed as a source of vibration of the vehicle driving on a straight road, with constant velocity. The vehicle is represented by equivalent spatial model which has seven degrees of freedom and, in which, masses, inertial moments and linear characteristics of elasto-damping elements (springs, dampers and tires) appear. Our specific goal here is to obtain an active suspension design i.e. control forces history coming from the road roughness. To do this in the second section we start with stability consideration more general than widely accepted Liapunov stability concept. Then in the third section we present nonlinear differential equations of motion of a motor vehicle as a multi-degree-of-freedom mechanical system. The subsequent quasi-linearization of these equations is necessary not only to simplify these differential equations but also to introduce internal as well as external resonances of the vehicle. The last section is devoted to optimal stabilization of the much simpler model with only two degrees of freedom.

2. Bertolino–Bakša stability concept

Consider a system of differential equations:

$$\dot{x} = A(t)x + f(t) + \varepsilon\Phi(t, x) \tag{2.1}$$

where are ($R \equiv (-\infty, \infty)$, $R^+ \equiv [0, \infty)$):

- $x \in R^n$ - the n-dimensional state vector,
- $A(t)$ - square matrix of the n-th order with elements $a_{ij} \colon R_+ \to R$,
- $t \in R^+$ - the time,
- $f \colon R^+ \to R^n$ - a vector function of the scalar variable,
- $\Phi \colon R^+ \times R^n \to R^n$ - another vector functions of the scalar variable and the state vector,
- ε - a small parameter,
- $\dot{x}$ - the time derivative of the state vector.

In the following, it is assumed that the vectorial equation (2.1) fulfils existence as well as uniqueness conditions.

The system (2.1) may represent, for instance, differential equations of motion of a mechanical system in the vicinity either of a stationary motion or of an equilibrium position. Therefore, its analysis should be interesting for an application in mechanics or elsewhere. Of course, there exists a whole sequence of other processes which can be adequately described by the vectorial equation (2.1).

In the linear approximation of (2.1) i.e.,

$$\dot{x}_{\text{lin}} = A(t)x_{\text{lin}} + f(t), \tag{2.2}$$

its Cauchy's solution satisfying the initial condition $t_0 = 0$, $x_{\text{lin}}(0) = x_0$ reads:

$$x_{\text{lin}}(t) = X(t)x_0 + \int_0^t X(t)X^{-1}(\tau)f(\tau)d\tau \tag{2.3}$$

where $X(t)$ is the normalized fundamental matrix of the homogeneous equation, obtained from (2.2) by putting $f(t) = 0$. From such a homogeneous equation, there follows: $\dot{X}(t) = A(t)X(t)$. Suppose that

$$x = x(t, x_0) \tag{2.4}$$

is the solution of the differential equation (2.1) which satisfies the initial condition $t_0 = 0$, $x(0) = x_0$. Taking into account the smallness of ε, it is reasonable to state the question how to estimate properly the discrepancy between the exact solution (2.4) and the function (2.3) on the real positive axis R^+. This question becomes of essential value if the method of small parameter should be applied in order to obtain the approximate solution of (2.1). Namely, by this method, an improvement of the solution (2.3) is performed and this procedure is correct only if in the proximity of (2.3) there exists the solution (2.4) of (2.1). Thus, the difference between (2.3) and (2.4) must be estimated. The task can be extended in the following way. Let

$$\hat{x} = x(t, \hat{x}_0) = \dot{x}(t) \tag{2.5}$$

be an arbitrary motion of the system (2.1) corresponding to a new initial condition $\hat{x}(0) = \hat{x}_0$ where $\|\hat{x}_0 - x_0\|$ is small (in other words, $\hat{x}_0$ belongs to the vicinity of x_0). Then, the extended task is to establish an estimation of discrepancy between (2.5) and (2.3) on R^+, As a special case, if $\hat{x}_0 = x_0$, the former problem of discrepancy between (2.3) and (2.4) on R^+ is obtained. Since the difference between x_0 and $\hat{x}_0$ may be understood as a perturbation of initial conditions, a comparison between (2.3) and (2.5) on R^+ is a stability problem. However, this is not a Liapounov stability problem because the function (2.3) is not a solution of (2.1). Nevertheless, a proper notion of stability corresponding to the stated extended problem can be defined and a suitable definition is found in [**4, 8**].

Definition 2.1 (M. Bertolino & A. Bakša). A function $\psi(t) = 0$, $\forall t \in R^+$ is an almost stable approximate solution of the equation:

$$\dot{x} = F(t, x), \tag{B-B1}$$

(with $x \in R^n$, $F\colon R^+ \times R^n \to R^n$) if to any arbitrary $\varepsilon \geqslant l > 0$ (l is a fixed number), there corresponds $\delta(\varepsilon, t_0) > 0$ such that any solution $x(t)$ of (B-B1) for which

$$\|x(0) - \psi(0)\| < \delta$$

holds, the inequality

$$\|x(t) - \psi(t)\| < \varepsilon$$

is satisfied for all $t \geqslant 0$.

According to this definition, the stated problem may be reformulated as: to examine conditions under which the function (2.3) is an almost stable approximate solution of the differential equation (2.1).

If (2.3) is taken as motion and the perturbation is denoted by $\zeta(\zeta \in R^n)$, then the disturbed motion (denoted by $\hat{x}$) is given by

$$\hat{x}(t) = X(t)x_0 + \int_0^t X(t)X^{-1}(\tau)f(\tau)d\tau + X(t)\zeta(t). \tag{2.6}$$

Since the perturbed motion has to fulfil (2.1), replacement of (2.6) into (2.1) gives the differential equation for perturbation:

$$\dot{\zeta} = \varepsilon X^{-1}(t)S(t,\zeta), \tag{2.7}$$

where

$$S(t,\zeta) \equiv \Phi\Big[t,\ X(t)x_0 + \int_0^t X(t)X^{-1}(\tau)f(\tau)d\tau + X(t)\zeta(t)\Big].$$

It is worth noting that, in general, a vanishing of perturbation, $\zeta = 0$, is not a solution of (2.7). In this way, the term $S(t,\zeta)$ in (2.7) acts as a function of permanent perturbations.

However, the stability in the sense of the above Bertolino–Bak'ša definition is not the same as the stability in the presence of permanent perturbations [13]. First, an almost stable approximate solution allows that there exists its vicinity $\|\zeta\| < l$ without disturbed motions (i.e., that it is not possible to approach an undisturbed motion by means of some disturbed motion in arbitrary way) and this is not the case in the definition of a stable solution in the presence of permanent perturbations. On the other hand, in order to have a motion stable in the presence of permanent perturbations, it is assumed that for $\|\zeta\| < \delta$, the absolute value of excitation $\|S(t,\zeta)\|$ is small enough for all t and this is not fulfilled in our case.

Now, in order to solve the problem stated above, let us find all arbitrary solutions of the equation (2.7).

It is a straightforward matter to see that for $\varepsilon = 0$, the solution $\zeta = c = \text{const.}$ follows, so that, in this case, $X(t)c$ represents the solution of the homogeneous part of the equation (2.2). According to the averaging method [11, 12], a solution of (2.7) could be wanted in the form

$$\zeta = v + \varepsilon w(v,t), \tag{2.8}$$

where a new variable v obeys the following differential equation:

$$\dot{v} = \varepsilon S_1(v) + \varepsilon^2 S_2(v) + \dots, \tag{2.9}$$

while functions $w, S_1, S_2, \dots$ should be additionally determined. If only the first approximation of (2.9) is considered, then (2.7) and (2.8) will give

$$\dot{v} = \varepsilon S_1(v), \tag{2.10}$$

where

$$S_1(v) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X^{-1}(t)\Phi[t,\ X(t)v]dt$$

and integration is performed only on the explicit t. A solution

$$\hat{v} = \hat{v}(t, \varepsilon) \tag{2.11}$$

of (2.10) represents an approximation of the corresponding exact solution of (2.7), being correct up to the order of ε on the interval $[0,\ 1/\varepsilon]$ (cf. [11, 12]).

The following question is now in order: is the approximation (2.11) valid also on the whole interval R^+? Paying attention to the fact that (2.7) may be approximately written in the form

$$\dot{v} = \varepsilon S_1(v) + \varepsilon O(\varepsilon),$$

it is clear that by rejecting the term $O(\varepsilon)$, a solution of the reduced equation is obtained with an error which can infinitely increase with time (when $t \to +\infty$). Therefore, it is necessary to examine conditions on the equation (2.7) in order to keep the error in the prescribed boundaries.

Suppose that:

1. the equation (2.10) has an asymptotically stable solution $v = 0$ and
2. the function

$$w(v, t) = \int_0^t \{X^{-1}(\tau)S[\tau, X(\tau)v] - S_1(v)\}d\tau \tag{2.12}$$

   as well as its derivatives are continuous and uniformly bounded on R^+ with respect to v and t.

Now, let $\zeta(t, \varepsilon)$ be a solution of (2.7). Then, replacing it into (2.8) and differentiating so obtained equality in the sense of the equation (2.7), we obtain:

$$\varepsilon X^{-1}(t)S[t,\ X(t)v] = \dot{v} + \varepsilon\frac{\partial w}{\partial v}\dot{v} + \varepsilon\frac{\partial w}{\partial t}$$

If in the above equation, w, which was undetermined until now, is replaced by (2.12), we get:

$$\dot{v} = \varepsilon S_1(v) - \varepsilon\frac{\partial w}{\partial v}|\varepsilon S_1(v) + O(v)|. \tag{2.13}$$

Due to the above assumptions, it is easy to show that for arbitrary $\Delta > 0$, there exists always $\delta > 0$, such that:

$$\left\|\frac{\partial w}{\partial v}[\varepsilon S_1(v) + O(v)]\right\| < \Delta$$

if $\|v\| < \delta$. Hence, (2.13) is a differential equation of a perturbed motion (with respect to $v = 0$) in the presence of permanent small perturbation

$$\lambda(v, t, \varepsilon) = \frac{\partial w}{\partial v}[\varepsilon S_1(v) + O(\varepsilon)]. \tag{2.14}$$

It is known (cf. [13]) that the asymptotically stable solution $v = 0$ of the autonomous system (2.10) is stable also in the presence of permanent perturbation (2.14). Consequently, if assumptions 1. and 2. are valid, the solution of (2.10) will approximate the corresponding solution of (2.7) with accuracy to the order of ε on the whole R^+.

Furthermore, it is easy to see that, from the stability of the solution $\zeta = 0$ of the equation (2.7), there follows the stability or the function (2.3) in the sense of Bertolino–Bakša definition under the condition that $X(t)$ is a uniformly bounded matrix on R^+.

3. Vehicle vibration model

Motor vehicle's motion on a rough road induces vibrations of the vehicle, the intensity of which depends on geometric characteristics of the road, design parameters of the vehicle and velocity of motion. Measurement of road roughness and mathematical modeling of its stochastic excitation were the subjects of many investigations (cf. [**3**, **5**]). In order to investigate vehicle vibrations due to road roughness during a straight-line translation, without the influence of the side wind, a spatial model of the vehicle is adopted with independent front suspension and dependent rear suspension. Vehicle's spatial position is determined by following Cartesian coordinate systems:

(1) a fixed, immobile coordinate frame,$Ox_0y_0z_0$ with O_{x0}-axis being situated at the intersection of the road plane with a vertical plane through the longitudinal axis of the vehicle, and Oy_0 axis situated at the intersection of the road plane with a vertical plane through the line connecting the centers of the rear wheels),
(2) mobile coordinate systems, $Cx_1y_1z_1$, attached to the vehicle's center of gravity (the axes of which are, at any instant, parallel to corresponding axes of the fixed system $Ox_0y_0z_0$) as well as mobile coordinate system, $Cxyz$, also attached to the vehicle's center of gravity, being fixed to the car body of the vehicle.

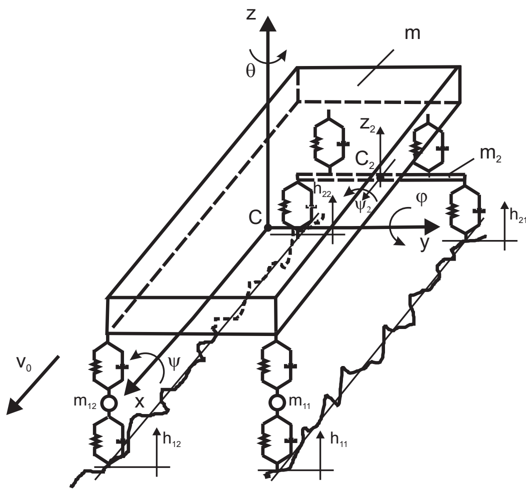


FIGURE 1. The spatial vehicle model as a system with 7 degrees of freedom (θ neglected)

In order to measure angular deflections of the $Cxyz$ system in relation to $Cx_1y_1z_1$ system, the angles ψ, φ, θ are chosen (around x, y and z axis, respectively). Due to the nature of the considered vehicle's motion, θ is taken to be equal to zero. The base unit vectors of systems $Cx_1y_1z_1$ are $i := \{i_1, i_2, i_3\}$ while for $Cxyz$ we have $e := \{e_1, e_2, e_3\}$. They are mutually related by means of:

$$e = Ri = \begin{bmatrix} \cos\varphi & \sin\psi\sin\varphi & \cos\psi\sin\varphi \\ 0 & \cos\psi & -\sin\psi \\ -\sin\varphi & \sin\psi\cos\varphi & \cos\psi\cos\varphi \end{bmatrix} i,$$

where R-is orthogonal Cartesian tensor of coordinate transformations. The chosen generalized coordinates of the model, depicted in the above figure, are: $q_1 = z$, $q_2 = \psi$, $q_3 = \varphi$, $q_4 = z_{11}$, $q_5 = z_{12}$, $q_6 = z_2$, $q_7 = \psi_2$.

3.1. Differential equations of motion. Differential equations of motion of the model (composed of the system of rigid bodies subjected to conservative forces, dissipative forces and other arbitrary non-conservative forces) can be expressed by Lagrange equations of the second kind. By making use of explicitely written expressions for kinetic and potential energies, generalized forces, relative deformation and velocity of relative deformation of elasto-damping elements, nonlinear differential equations of motion of the vehicle model can be written as follows:

- for vertical vibration of the suspended mass (z)

$$\begin{aligned} m\ddot{z} &+ 2(b_1+b_2)\dot{z} + 2(c_1+c_2)z + 2(b_2b - b_1a)\cos\varphi\dot{\varphi} - b_1\dot{z}_{11} \\ &- c_1z_{11} - b_1\dot{z}_{12} - c_1z_{12} - 2b_2\dot{z}_2 - 2c_2z_2 + 2(c_2b - c_1a)\sin\varphi = 0, \end{aligned} \tag{3.1}$$

- for vertical vibration of the front left wheel (z_{11})

$$\begin{aligned} -b_1\dot{z} &- c_1z - b_1s_1\cos\psi\cos\varphi\dot{\psi} + b_1(a\cos\varphi + s_1\sin\psi\cos\varphi)\dot{\varphi} + m_{11}\ddot{z}_{11} \\ &+ b_1\dot{z}_{11} + (c_1+c_{p1})z_{11} + c_1a\sin\varphi - c_1s_1\sin\psi\cos\varphi - c_{p1}h_{11} = 0, \end{aligned} \tag{3.2}$$

- for vertical vibration of the front right wheel (z_{12})

$$\begin{aligned} -b_1\dot{z} &- c_1z + b_1s_1\cos\psi\cos\varphi\dot{\psi} + b_1(a\cos\varphi - s_1\sin\psi\cos\varphi)\dot{\varphi} + m_{12}\ddot{z}_{12} \\ &+ b_1\dot{z}_{12} + (c_1+c_{p1})z_{12} + c_1a\sin\varphi + c_1s_1\sin\psi\cos\varphi - c_{p1}h_{12} = 0, \end{aligned} \tag{3.3}$$

- for vertical vibration of the rear suspension system (z_2)

$$\begin{aligned} -2b_2\dot{z} &- 2c_2z - 2b_2b\cos\varphi\dot{\varphi} + (m_{21}+m_{22}+m_z)\ddot{z}_2 + 2b_2\dot{z}_2 \\ &+ 2(c_2+c_{p2})z_2 - 2c_2b\sin\varphi - c_{p2}h_{21} - c_{p2}h_{22} = 0, \end{aligned} \tag{3.4}$$

- for rotation of the car body around longitudinal axis (ψ)

$$\begin{aligned} J_x\ddot{\psi} &+ 2(b_1s_1^2 + b_2s_0^2)\cos^2\psi\cos^2\varphi\dot{\psi} + (J_y - J_z)\sin\psi\cos\psi\dot{\varphi}^2 \\ &- 2(b_1s_1^2 + b_2s_0^2)\sin\psi\cos\psi\sin\varphi\cos\varphi\dot{\varphi} - b_1s_1\cos\psi\cos\varphi\dot{z}_{11} \\ &- c_1s_1\cos\psi\cos\varphi z_{11} + b_1s_1\cos\psi\cos\varphi\dot{z}_{12} \\ &+ c_1s_1\cos\psi\cos\varphi z_{12} - 2b_2s_0^2\cos\psi\cos\varphi\cos\psi_2\dot{\psi}_2 \\ &+ 2(c_1s_1^2 + c_2s_0^2)\sin\psi\cos\psi\cos^2\varphi + 2c_2s_0^2\cos\psi\cos\varphi\sin\psi_2 = 0, \end{aligned} \tag{3.5}$$

- for rotation of the car body around transverse axis (φ)

$$
\begin{aligned}
(3.6)\quad & (b_2 b - b_1 a)\cos\varphi\dot{z} + 2(c_2 b - c_1 a)\cos\varphi z \\
& - 2(b_1 s_1^2 + b_2 s_0^2)\sin\psi\cos\psi\sin\varphi\cos\varphi\dot{\psi} \\
& - 2(J_y - J_z)\sin\psi\cos\psi\dot{\psi}\dot{\varphi} + \left[J_y - (J_y - J_z)\sin^2\psi\right]\ddot{\varphi} \\
& + 2\left[\left(b_1 a^2 + b_2 b^2\right)\cos^2\varphi + \left(b_1 s_1^2 + b_2 s_0^2\right)\sin^2\psi\sin^2\varphi\right]\dot{\varphi} \\
& + b_1(a\cos\varphi + s_1\sin\psi\sin\varphi)\,\dot{z}_{11} + c_1(a\cos\varphi + s_1\sin\psi\sin\varphi)\,z_{11} \\
& + b_1(a\cos\varphi - s_1\sin\psi\sin\varphi)\,\dot{z}_{12} + c_1(a\cos\varphi - s_1\sin\psi\sin\varphi)\,z_{12} \\
& - 2b_2 b\cos\varphi\dot{z}_2 - 2c_2 b\cos\varphi z_2 - 2b_2 s_0^2\sin\psi\sin\varphi\cos\psi_2\dot{\psi}_2 \\
& + 2(c_1 a^2 + c_2 b^2)\sin\varphi\cos\varphi - 2(c_1 s_1^2 + c_2 s_0^2)\sin^2\psi\sin\varphi\cos\varphi \\
& - 2c_2 s_0^2\sin\psi\sin\varphi\sin\psi_2 = 0,
\end{aligned}
$$

- for rotation of the rear suspension system around longitudinal axis (ψ_2)

$$
\begin{aligned}
(3.7)\quad & 2b_2 s_0^2\cos\psi\cos\varphi\cos\psi_2\dot{\psi} - 2b_2 s_0^2\sin\psi\sin\varphi\cos\psi_2\dot{\varphi} \\
& + \left[(m_{21} + m_{22})\,s_2^2 + J_{z\psi 2}\right]\ddot{\psi}_2 + 2b_2 s_0^2\cos^2\psi_2\dot{\psi}_2 + 2c_2 s_0^2\sin\psi\cos\varphi\cos\psi_2 \\
& + 2(c_2 s_0^2 + c_{p2} s_2^2)\sin\psi_2\cos\psi_2 - c_{p2} s_2(h_{21} - h_{22})\cos\psi_2 = 0.
\end{aligned}
$$

Herein we have

m – vehicle body mass,
m_{11}, m_{12} – front left/right wheel mass, respectively,
m_{21}, m_{22} – rear left/right wheel mass,
m_2 – rear twist beam suspension mass,
h_{11}, h_{12} – front left/right wheel excitation due to road roughness, respectively,
h_{21}, h_{22} – rear left/right wheel excitation due to road roughness,
c_1, c_2 – stiffness coefficient of the front/rear suspension,
c_{p1}, c_{p2}-stiffness coefficient of the front/rear wheel,
b_1, b_2 – damping coefficient of the front/rear suspension,
b_{p1}, b_{p2}-damping coefficient of the front/rear wheel,
J_x, J_y, J_z-vehicle body moments of inertia about x, y and z axis,
$J_{z\psi 2}$ – rear suspension moment of inertia about its longitudinal axis,
l – wheelbase,
a, b – longitudinal distance of the center of gravity from the front/rear axle,
$2s_1, 2s_2$ – front/rear wheel track,
$2s_0$ – lateral distance between rear shock absorbers.

3.2. Quasi-linearization of the differential equations of motion. The system of differential equations (3.1)–(3.7) can be written in a short form as [5] (here, Einsteins convention for summation over repeted indices is applied):

$$
(3.8)\qquad \tilde{M}_{ij}(q)\ddot{q}_j - \tilde{M}_{ijk}(q)\dot{q}_j\dot{q}_k = -\tilde{C}_i(q) - \tilde{B}_{ij}(q)\dot{q}_j + \tilde{D}_i(q,h),
$$

where: $i, j, k \in \{1, \ldots, 7\}$ and $\tilde{M}_{ij}(q)$, $\tilde{M}_{ijk}(q)$ are inertial coefficients dependent on generalized coordinates, $\tilde{C}_i(q)$ are generalized conservative forces as non-linear functions of coordinates, $\tilde{B}_{ij}(q)$ are variable coefficients of the viscous resistance of dampers as functions of coordinates and $\tilde{D}_i(q, h(t))$ are general non-autonomous excitation forces inducing parametric vibrations dependent on generalized coordinates, q_i, as well as on road roughness parameters, $h_{\alpha\beta}(t)$, $(\alpha, \beta \in \{1, 2\})$. By multiplication of the expression (3.8) with the matrix $\left[\tilde{M}_{ij}\right]^{-1}$, the system is reduced to:

$$\ddot{q}_i + \bar{B}_{ij}(q)\dot{q}_j + \bar{C}_i(q) = \bar{M}_{ijk}(q)\dot{q}_j\dot{q}_k + \bar{D}_i(t, q), \tag{3.9}$$

Solving the system of non-linear differential equations (3.9) is a very complicated task, because non-linear functions of generalized coordinates and time appear in the above tensors $\bar{B}, \bar{C}, \bar{M}, \bar{D}$.

For this purpose, a development of their elements into power series is applied in the sequal. These series are: for non-autonomous excitation forces $\bar{D}_i(t, q) = \bar{Q}_i(t) + \bar{P}_i(t, q)$, where $\bar{Q}_i(t) = \bar{d}_{ij}\bar{h}_j$ is a vector of non-autonomous excitation forces, and $\bar{P}_i(t, q) = d_{ijkl}h_j(t)q_kq_l$ is a matrix of non-autonomous forces which induce parametric vibration of vehicle; for inertial coefficients of the third order $\bar{M}_{ijk}(q) = \bar{m}_{ijkl}q_l$; for conservative force coefficients $\bar{C}_i(q) = \bar{c}_{ij}q_j$ and for viscous resistance coefficients as $\bar{B}_{ij}(q) = \bar{b}_{ij} + \bar{b}_{ijkl}q_kq_l$. Non-linear equations now have a simplified form:

$$\ddot{q}_i + \bar{b}_{ij}\dot{q}_j + \bar{c}_{ij}q_j - \bar{Q}_i(t) = \bar{R}_i(q, \dot{q}) + \bar{P}_i(t, q),$$

where $\bar{R}_i(q, \dot{q}) = \bar{m}_{ijkl}\dot{q}_j\dot{q}_kq_l - \bar{b}_{ijkl}\dot{q}_jq_kq_l$ are non-linear autonomous forces [1].

By introducing

$$x_i = q_i, \quad x_{i+7} = \dot{q}_i, \qquad (i \in \{1, \dots, 7\}),$$

seven differential equations of the second order have been transformed to fourteen differential equations of the first order:

$$\dot{x} - Ax - Q(t) = R(x) + P(t, x). \tag{3.10}$$

Matrix A is directly connected to the fundamental matrix of the system (cf. [16]), while autonomous non-linear generalized forces and non-autonomous generalized forces (inducing parametric vibration) are represented by 14-dimensional column vectors $R(x)$, $P(t, x)$, respectively (with previously performed transformations $q_i = q_i(x)$, $\dot{q}_j = \dot{q}_j(x)$).

Remark 3.1 (Conditions for quasilinearization). The above procedure is possible if Liapunov stability condition for stationary motion of the vehicle is satisfied: if $\forall\epsilon > 0$, $\exists\delta(\epsilon) > 0$, such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon$.

Free vibrations of the linearized model are determined by the system of differential equations:

$$\dot{x}^{(h)}_{\text{lin}} - A\,x^{(h)}_{\text{lin}} = 0. \tag{3.11}$$

This system is linear, so its solution is assumed in the form: $x^{(h)} = Ce^{\lambda t}$. The system of homogenous linear equations in C_i, $(i = 1, 2, \dots, 14)$ has non-trivial solutions only if: $|\lambda 1 - A| = \sum_{k=0}^{14} a_k\lambda^{14-k} = 0$. Calculated roots of its characteristic

[1] Values of all listed coefficients, i.e., $b_{ij}, \bar{b}_{ijkl}, m_{ij}, \bar{m}_{ijkl}, d_{ij}, \bar{d}_{ijkl}$ are given explicitely in [6] for Yugo Florida and driving conditions considered above.

polynomial meet the requirement for the stability of motion, because all real parts of the roots are negative. Each root of the characteristic polynomial determines one mode of vibration and the general solution can be written in the form:

$$x_{\rm lin}^{(h)} = K\xi_{\rm lin}^{(h)}, \tag{3.12}$$

where: $x_{\rm lin}^{(h)}$ is a column state vector dependent on initial conditions for generalized coordinates and velocities, $\xi_{\rm lin}^{(h)}$ is a column vector of binormal coordinates of free damped vibrations and K is the modal matrix of the system. Multiplying the expression (3.11) by K^{-1} from the left side and considering (3.12), an expression for forced vibration is obtained:

$$\dot{\xi}_{\rm lin}^{(h)} - \Lambda\xi_{\rm lin}^{(h)} = f(t), \tag{3.13}$$

where $f(t) = K^{-1}Q(t) = \hat{d}h(t)$, and $\hat{d} = [\bar{d}_{i\,j}]$ is a constant complex matrix, whereas Λ is the diagonal matrix of eigenvalues. The general solution of this linear differential equation has the form (2.3).

Introduction of the transformation of coordinates (3.12) into the nonlinear equation (3.10) and multiplication by K^{-1} from the left side leads to:

$$K^{-1}(K\dot{\xi} - AK\xi) = K^{-1}(R(K\xi) + P(t, K\xi) + Q(t)).$$

Since the modal matrix makes the matrix A diagonal, we have:

$$\dot{\xi}_k - \Lambda_k\xi_k = f_k(t) + \varepsilon\Phi_k(t,\xi), \tag{3.14}$$

(without summation) where: Λ is a diagonal matrix of the roots of the characteristic polynomial, $f_i(t)$ is a column vector of excitation forces related to the linearized model, and $\varepsilon\Phi(t,\xi)$ is a column vector including non-linear autonomous generalized forces as well as generalized forces inducing parametric vibration. For assumed motion, the elements of the column vector $\varepsilon\Phi(t,\xi)$ are in the form of $\varepsilon\Phi_i(t,\xi) = \varepsilon\alpha_{ijkl}\xi_j\xi_k\xi_l + \varepsilon\psi_{ijk}(t)\xi_j\xi_k$. Here, autonomous constant coefficients are $\alpha_{abcd} = K^{-1}_{(a)i}r_{ijkl}K_j^{(b)}K_k^{(c)}K_l^{(d)}$, whereas $\psi_{abc}(t) = K^{-1}_{(a)i}d_{ijkl}h_j(t)K_k^{(b)}K_l^{(c)}$ are parametric coefficients, $K_i^{(j)}$ being the i-th column and the j-th row element of the modal matrix K.

3.3. Negligible parametric excitation. In this case $P(t, K\xi) = 0$. Let a solution for ξ_i is assumed to be approximately given by a power series in the small parameter ε (cf. [11]):

$$\xi_i(t) = \xi_i^{(0)}(t) + \varepsilon\xi_i^{(1)}(t) + \varepsilon^2\xi_i^{(2)}(t) + O(\varepsilon^3). \tag{3.15}$$

If it is replaced into (3.14), the grouping of the members of different powers in the small parameter, ε, gives a set of the following differential equations:

$$\begin{aligned} &\dot{\xi}_i^{(0)} - \lambda_{(i)}\xi_i^{(0)} = f_i(t), \\ &\dot{\xi}_i^{(1)} - \lambda_{(i)}\xi_i^{(1)} = \alpha_{ijkl}\xi_j^{(0)}\xi_k^{(0)}\xi_l^{(0)}, \\ &\dot{\xi}_i^{(2)} - \lambda_{(i)}\xi_i^{(2)} = \alpha_{ijkl}\big(\xi_j^{(1)}\xi_k^{(0)}\xi_l^{(0)} + \xi_j^{(0)}\xi_k^{(1)}\xi_l^{(0)} + \xi_j^{(0)}\xi_k^{(0)}\xi_l^{(1)}\big), \end{aligned} \tag{3.16}$$

which are solved in the following order: $\xi_i^{(0)}, \xi_i^{(1)}, \xi_i^{(2)}$ by standard methods (cf. [11]). In such systems it is of interest to examine the extent to which the effect of the perturbation force corresponding to one coordinate is transferred to other coordinates. Namely, it can happen that such a perturbation force $f_i(t)$ causes a considerable oscillation of some other coordinate ξ_j, $(j \neq i)$ whereas perturbation of ξ_i becomes smaller. This phenomenon is known as the "*spatial instability*" of the system. Obviously, this means a redistribution of energy given to the system by this force. This may happen if the connection between the dynamic system coordinates is such that the total energy of the system is transferred to only one (or more, but not all) coordinates. The analysis of such a spatial instability can only be performed on non-linear models, since by the linearization $\Phi_k = 0$, $k \in \{1, \dots, 14\}$ and the interconnection among the coordinates that allows this instability ceases. The study of the spatial instability, is usually performed in two stages:

(1) The conditions under which, in a given dynamic system, spatial instability can occur.
(2) The oscillations if such conditions of spatial instability are fulfilled.

Since this type of instability is undesirable, the task is practically reduced to the fact that, based on the results of the first stage, the geometrical and dynamic characteristics of the vehicle are chosen to prevent it.

For simplicity, suppose that only left front wheel meets an uneveness of a general form, whereas the others are still on a flat roadway. Then we would have (with new notations: $\chi_1 \equiv h_{11}$, $\chi_2 \equiv h_{12}$, $\chi_3 \equiv h_{21}$, $\chi_4 \equiv h_{22}$):

$$\chi_1(t) = \sum_{\nu=1}^{\infty} A_\nu \sin \nu\Omega t, \qquad \chi_k = 0 \quad (k = 2, 3, 4),$$

where the constants A_ν and Ω define the given Fourier series. In this case $f_i(t)$ and $P_i(t, \xi)$ simplify into:

$$f_i(t) = \gamma_{i1} \sum_{\nu}^{\infty} A_\nu \sin \nu\Omega t,$$

$$P_i(t, \xi) = \varepsilon \sum_{j}^{14} \sum_{\nu}^{\infty} B_{\nu ij} \xi_j \sin \nu\Omega t + \varepsilon \sum_{j,k}^{14} \sum_{\nu}^{\infty} B_{\nu ijk} \xi_j \xi_k \sin \nu\Omega t,$$

with $B_{\nu ij} = A_\nu \pi_{i1j}$, $B_{\nu ijk} = A_\nu\ \pi_{i1jk}$,

The case $\boldsymbol{\varepsilon = 0}$. The linearized equations (3.13) have now the solutions:

$$\xi_k = C_k e^{\lambda_k t} + \sum_{\nu}^{\infty} P_{k\nu} \sin(\nu\Omega t + \psi_{\nu k}), \qquad (k = 1, \dots, 14), \tag{3.17}$$

with eigenvalues of the characteristic polynom $\lambda_k = \alpha_k + i\beta_k$, $\lambda_{k+7} = \alpha_k - i\beta_k$, $(k = 1, \dots, 7)$, $(\alpha_k < 0)$. Here

$$P_{k\nu} = \frac{\gamma_{k1} A_\nu}{\sqrt{(\lambda_k^2 + \nu^2\Omega^2)}}, \quad \psi_{\nu k} = \arctan\left(\frac{\nu\Omega}{\lambda_k}\right),$$

are the linear amplitude and linear phase of the ν-th sine function. In other words, we write $C_i \exp \lambda_i t = C_i \exp(\alpha_i + i\beta_i)t \equiv s_i(t) \exp i\beta_i t$, where the monotonuously decreasing amplitudes are denoted by $s_i(t)$.

The case $\varepsilon \neq 0$. We again seek the solution of the nonlinear differential equations (3.13) in the form (3.17). Then, (3.13) transform into:

$$\dot{s}_k - \alpha_k s_k = e^{-i\beta_k t}\Phi_k(\varepsilon, t, s_1, \ldots, s_{14}), \qquad (k = 1, \ldots, 14). \tag{3.18}$$

Since these differential equations depend on ε explicitly, we try to find their solution in the form:

$$s_i = s_i^0 + \varepsilon s_i^1 + \varepsilon^2 s_i^2 \ldots. \tag{3.19}$$

The lowest order part of the solution, s_i^0, is found from (3.18) after its right hand side is averaged by the method of Krylov-Bogoliubov by the integration in time domain:

$$\bar{\Phi}_k = \lim_{T\to\infty} \int_0^T e^{-i\beta_k t}\Phi_k(\varepsilon, t, s)dt.$$

Time appears explicitly and the other variables during integration are kept fixed. Let us suppose that "*internal*" (i.e. $\beta_k \neq \beta_l$, $k \neq l$, $\forall k, l \in \{1, \ldots, 7\}$) as well as "*external*" resonances (i.e. $\beta_k \neq \nu\Omega\forall k, \nu$, $(k \in \{1, \ldots 14\}$, $\nu \in \{1, 2, \ldots\})$. Then, are absent we arrive at:

$$\begin{aligned}
\bar{R}_i &= \lim_{T\to\infty} \int_0^T e^{-i\beta_i t} R_i(\varepsilon, s)dt = \varepsilon s_i \left(A_i + \sum_j^7 \bar{\alpha}_{iijj+7} s_j s_{j+7} \right), \\
\bar{P}_i &= \lim_{T\to\infty} \int_0^T e^{-i\beta_i t} P_i(\varepsilon, s, t)dt = \varepsilon s_i B_i, \\
A_i &= \frac{1}{2} \sum_{j,k}^{14} \bar{\alpha}_{iijk} \sum_\nu^\infty P_{j\nu} P_{k\nu}, \\
B_i &= \frac{1}{2} \sum_j^{14} \sum_\nu^\infty (B_{\nu iij} + B_{\nu iji}) P_{j\nu} \cos\psi_{\nu j},
\end{aligned} \tag{3.20}$$

$$\bar{\alpha}_{ijkl} = \alpha_{ijkl} + \alpha_{iklj} + \alpha_{iljk} + \alpha_{ikjl} + \alpha_{ijlk} + \alpha_{ilkj} \equiv 6\alpha_{i(jkl)}$$

with constants dependent on vehicle's features and road uneveness. Let us replace right hand sides of (3.18) by their averaged values (3.20). The result of this averaging is the following set of nonlinear differential equations in variable amplitudes

$$\dot{s}_k - s_k \left[\alpha_k + \varepsilon(A_k + B_k) + \sum_j^7 \bar{\alpha}_{kkjj+7} s_j s_{j+7} \right] = 0, \qquad (k \in \{1, \ldots, 14\})$$

whose stability should be investigated.

However, by the theorem of stability of nonlinear systems [13], if the linearized system is asymptotically stable, then the corresponding motion of the nonlinear system is at least stable. In order to have the corresponding linearized set of equations

$$\dot{s}_k - s_k[\alpha_k + \varepsilon(A_k + B_k)] = 0, \qquad (k \in \{1, \ldots, 14\})$$

asymptotically stable, it is necessary that all the coefficients s_k, $k \in \{1, \dots, 14\}$ be negative, i.e. that

$$\alpha_k + \varepsilon(A_k + B_k) < 0, \qquad (k = 1, \dots, 14).$$

If these conditions are fulfilled, then the method of small parameter may be applied by the series (3.19). The determination of higher order terms follows procedure the same as in (3.15)–(3.16).

REMARK 3.2 (Internal resonance). The internal resonance appears if anyone of the following equalities ($\forall j, k, l, m \in \{1, \dots, 7\}$)

$$\lambda_j + \lambda_k - \lambda_l = \alpha_j + \alpha_k - \alpha_l + i(\beta_j + \beta_k - \beta_l) = 0,$$
$$\lambda_j + \lambda_k + \lambda_l - \lambda_m = \alpha_j + \alpha_k + \alpha_l - \alpha_m + i(\beta_j + \beta_k + \beta_l - \beta_m) = 0$$

holds (cf. [5]). Then the spatial instability may take place. Thus, we may conclude that the development (3.19) could be made

(1) either around solution of the nonlinear differential equations (2.1) under Liapunov stability condition (cf. remark 3.1)
(2) or around the simplest (and most comfortable) motion $x(t) = 0$ under Bertolino-Bakša stability condition (cf. definition 2.1).

3.4. General case. Let us consider, now, that the parametric excitation $P(t, \xi)$ is also taken into account. The differential equations of motion (3.1)–(3.7) are solved without quasilinearization by the code SIMULINK of MATLAB, able to simulate online integration as well as differentiation of individual functions. The parameters for the vehicle Yugo Florida (cf. [6]) are: the small parameter $\varepsilon = 0.01$, the amplitude of the sinusoidal road wave of $A = 0.01$ m, the speed of the vehicle of $v_0 = 10$ m/s and the wave length of the road $\Lambda = 1$ m. The real motion of the 7D model over and after an uneveness situated not perpendicularly on undisturbed velocity vector is affected by $h_{12}(t) = h_{11}(t + \tau_0)$ as well as $h_{22}(t) = h_{21}(t + \tau_0)$ with a constant delay τ_0 between the first and the second peak as well as between the third and the fourth peak in fig. 2. Plotting of the excitating uneveness with all the seven coordinates as functions of time is depicted in the figs. 2 to 6. Looking

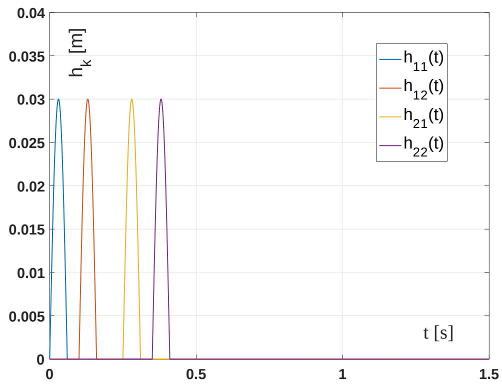
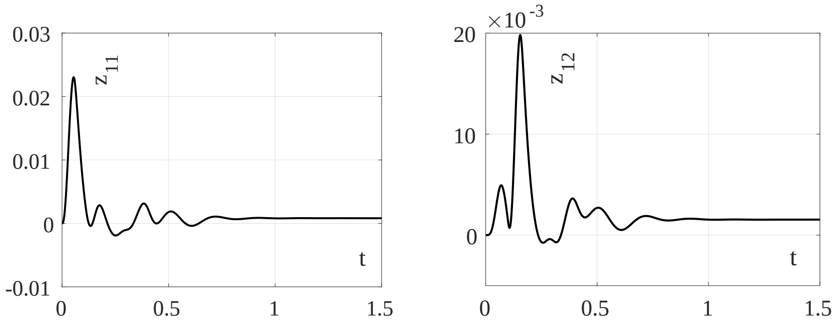


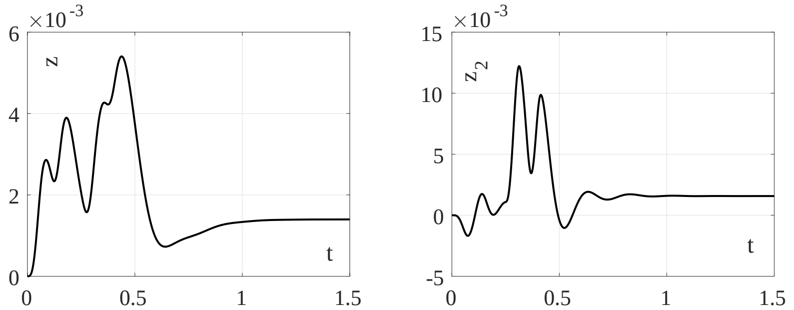
FIGURE 2. Excitations on all the wheels

at the figs. 3 and 4 of the translational coordinates z, z_{11}, z_{12}, z_2 we see a peculiar behavior that their perturbations do not vanish, but remain constant after crossing the unevenness by rear wheels. According to the monograph [18], nonlinear vibration may cause a shift of the stable equilibrium position of the system which is often called *vibrational shift*. It is important to note here that in the light of Bertolino-Bakša stability concept such a possibility is permitted.



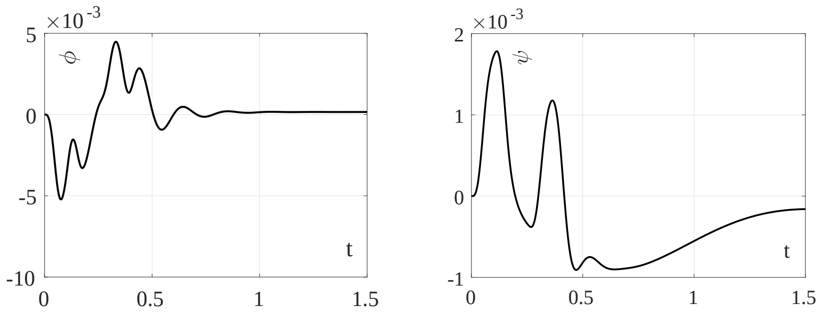
(a) Left wheel (b) Right wheel

FIGURE 3. Vertical displacements of front wheels



(a) Gravity center of the car body (b) Gravity center of the H-suspension bary

FIGURE 4. Vertical displacements



(a) Angle around longitudinal axis (b) Angle around transverse axis

FIGURE 5. Car rotation angles

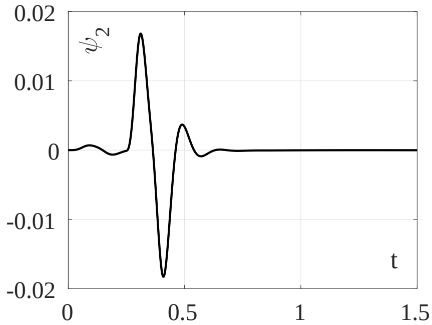


FIGURE 6. Rotation angle of the rear system of torsion beam suspension around longitudinal axis

4. Optimal stabilization

In this section we analyze a linearized model of a vehicle moving by rectilinear translation with velocity v which meets, first by front and then by rear wheels, an unevenness on a horizontal roadway. The motion of the simplified twodimensional model is shown in the fig. 7. The excitation (upper half of the sine wave) is the same as for the 7D model but perpendicular to velocity of the vehicle (with $h_{12}(t) = h_{11}(t)$ as well as $h_{21}(t) = h_{22}(t)$ with $\tau_0 = 0$ - cf. fig. 2). Our task is to perform an optimal stabilization of the motion after the come-down of all the wheels to be finished within a finite time interval. Differential equations of the motion for such a 2D model read:

$$\dot{x} = Ax + Bu + C(t), \tag{4.1}$$

where are:

$$x = [z\ \phi\ \dot{z}\ \dot{\phi}]^T,$$

$$u = [u_1\ u_2]^T,$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(c_1+c_2)/m & -(ac_1-bc_2)/m & -(b_1+b_2)/m & -(ab_1-bb_2)/m \\ -(ac_1+bc_2)/J & -(a^2c_1-b^2c_2)/J & -(ab_1-bb_2)/J & -(a^2b_1+b^2b_2)/J \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/m & -1/m \\ -1/J & -1/J \end{bmatrix}, \qquad C = \begin{bmatrix} 0 \\ 0 \\ (c_1z_A + c_2z_B + b_1\dot{z}_A + b_2\dot{z}_B)/m \\ (ac_1z_A - bc_2z_B + ab_1\dot{z}_A - bb_2\dot{z}_B)/J \end{bmatrix}.$$

We note here that the state variables are: the vertical displacement, the angle of deflection from horizontal plane as well as their time derivatives, whereas the control is realized by the functions u_1 and u_2. Matrices A and B are constant while some elements of the matrix $C(t)$ are functions of time taking into account that

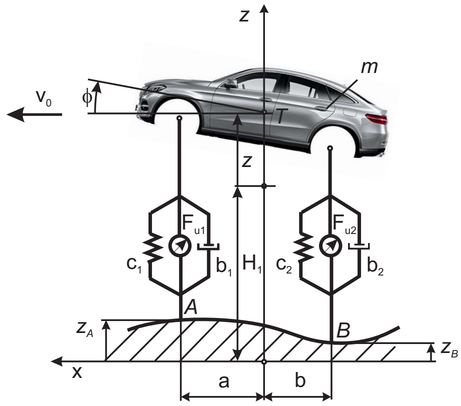


FIGURE 7. 2D-model

here we analyze the case when the unevenness has sine function shape depicted in the fig. 2 (perpendicular to velocity vector).

Here t_{A1} and t_{A2} are the instants of come-up and come-down of front wheels on uneveness, whereas t_{B1} and t_{B2} are the same instants for rear wheels.

Before ascension of front wheels onto the unevenness, the state vector X is null such that by solving Cauchy's problem (4.1) within the time interval $[t_{A1}, t_{B2}]$ it is possible to determine state of the system at the instant t_{B2}

$$x(t_{B2}) = [z(t_{B2})\phi(t_{B2})\dot{z}(t_{B2})\dot{\phi}(t_{B2})]^T. \tag{4.2}$$

We perform the optimal stabilization for the subsequent finite interval T in such a way that the functional

$$\frac{1}{2}\int_{t_{B2}}^{T}(x^TQx + u^TRu)dt$$

is minimized, where are

$$Q = \lambda\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & H_1^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \qquad R = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

The role of the parameter λ is to change relative magnitude of the corresponding terms within the functional.

This case belongs to the class of linear systems with quadratic functional on a finite time interval (cf. [**15**]). It is necessary to find optimal controls and corresponding optimal trajectories.

In order to apply the Pontriagin maximum principle and having in mind that in the subsequent motion the vector C vanishes, we need Hamilton function

$$H = \tfrac{1}{2}(x^TQx + u^TRu) + p^T(Ax + Bu),$$

where p is the co-state vector, and the adjoint system of equations has the form:

$$\dot{p} = -\partial_x H = -Qx - A^T p. \tag{4.3}$$

Optimal controls follow from the minima of Hamilton function with regard to the control vector, i.e.

$$0 = \partial_u H = Ru + B^T p.$$

Therefore:

$$u = -R^{-1} B^T p. \tag{4.4}$$

Replacement of the above relation into starting system (4.1) and the adjoint system (4.3) leads to the two points boundary problem. Here to the conditions (4.2) it is necessary to add transversality conditions on the right side:

$$p(T) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{4.5}$$

obtained from the fact that in this problem state variables are not given at the end instant of time.

In the theory of optimal regulators it is known that such two points boundary problem is reduced to solving of two Cauchy problems having unique solutions. Hence, the solution (4.4) is also unique and optimal.

Here the solution may be sought in the form

$$p = Lx \tag{4.6}$$

with L being an unknown symmetric quadratic matrix. Now, from (4.1), (4.3) and (4.4) and by differentiating (4.6) we arrive at the relationship

$$(\dot{L} + LA - LBR^{-1}B^T L + Q + A^T L)x = 0$$

which produces the matrix Riccati differential equation

$$\dot{L} + LA - LBR^{-1}B^T L + Q + A^T L = 0 \tag{4.7}$$

with ending condition $L(T) = 0$ being a consequence of (4.5) and (4.6).

The matrix differential equation (4.7), due to symmetry of L leads to 10 scalar nonlinear ordinary differential equations. The explicit solution is not possible and by the backward numerical integration in the interval $[t_{B2}, T]$ we get the matrix L as a function of time. In the next fig. 8 history of the element $L_{14}(t)$ is depicted. The other elements of $L(t)$ have similar forms.

Finally, with thus determined $L(t)$, and as an another Cauchy problem, on the basis of (4.1), (4.4) and (4.6) we have to solve the matrix differential equation

$$\dot{x} = Ax - BR^{-1}B^T Lx$$

for the interval $[t_{B2}, T]$ with initial conditions (4.2).

In the subfigures of fig. 9 vertical displacement as well as rotation angle histories are shown. Red lines stand for optimal stabilization whereas the blue lines are for motion without control forces.

Concerning time shifts between front and rear axle, the following values are taken:

$$\tau_{A1} = 0, \qquad \tau_{A2} = \frac{\Lambda}{2v_0} = 0.05\,\text{s},$$

$$\tau_{B1} = \tau_{A1} + \frac{a+b}{v_0} = 0.25\,\text{s}, \qquad \tau_{B2} = \tau_{A2} + \frac{a+b}{v_0} = 0.3\,\text{s},$$

where translational speed is $v_0 = 10\,\text{m/s}$, uneveness length $\Lambda = 1\,\text{m}$ and $a + b = 2.5\,\text{m}$.

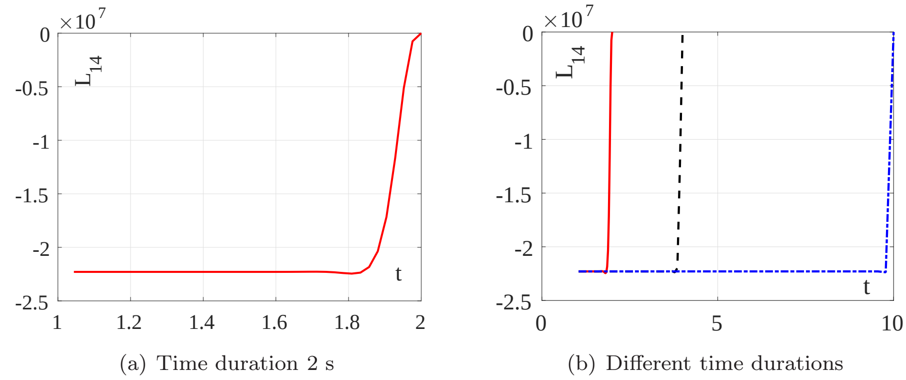


FIGURE 8. The history of L_{14}

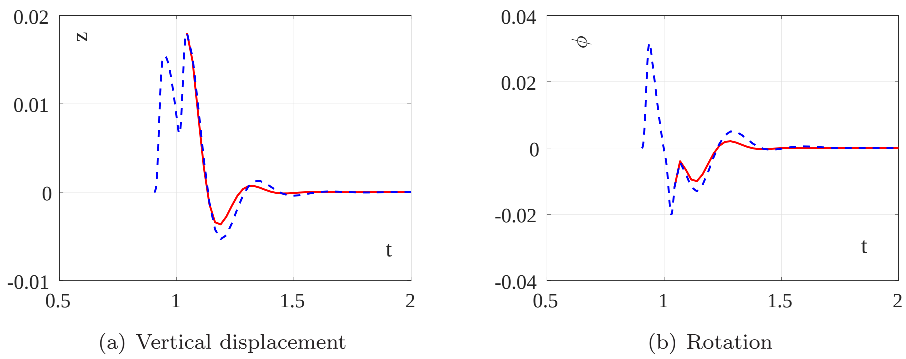
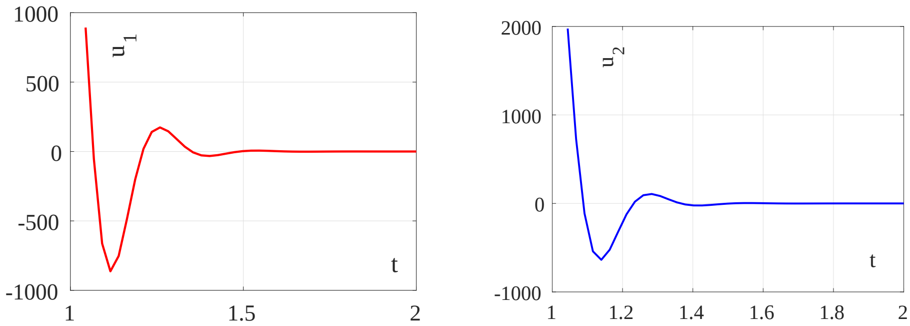


FIGURE 9. Histories of the centre of gravity vertical displacement as well as rotation angle

In this way (4.4) and (4.6) permit finding of the control forces which make possible the optimal motion.

REMARK 4.1. If the time interval is very long, i.e. may be taken as infinite, the control leading to optimal stabilization becomes considerably simplified. This case is given in detail in the book [16] where it is shown that the matrix L in such



(a) U1-force (b) U2-force

FIGURE 10. Histories of control forces

a case is constant. The Riccati equation (4.7) is reduced to the set of algebraic equations

$$LA - LBR^{-1}B^TL + Q + A^TL = 0.$$

Their solution gives us L which as a consequence leads to a very simple control

$$u = KX, \qquad K = -R^{-1}B^TL.$$

For determination of the Kalman matrix the code MATLAB has the function LQR. The solutions by Mathematica and Matlab are the same. Cauchy's problems are solved here by the routine NDSolve within the code MATHEMATICA leading to the next figures.

5. Conclusion

On the basis of the theoretical analysis given in the first section a very important, in stability of motor vehicles, problem is considered. The perturbation of stationary rectilinear translation is caused by roadway uneveness of the sinusoidal shape.

Nonlinear differential equations of oscilatory motion are solved by the sub-code SIMULINK of the code MATLAB without any stabilization. Especially uncomfortable appearance of rotations around transverse and longitudinal axis of the vehicle is found (cf. fig. 5).

Analisis of spatial (in)stability with internal resonances is made by a simplification method of the original differential equations called quasilinearization. Such (in)stability is analyzed by Liapunov as well as Bertolino-Bakša stability condition. The second is much simpler since we need to solve the linear differential equations of steady rectilinear translation. It is important that based on the second concept it is possible to build in a vehicle a very simple local computer control which would give commands to actuating forces aimed to return the disturbed motion to the steady translation. It must be noted that a complete linearization of the differential equations of motion would lead to vanishing of the spatial (in)stability. Thus, it cannot by analized by means of the linearized model.

In the last section, in order to find such actuating forces we apply optimal stabilization method by making use of Pontriagin principle. A special case of long stabilization time is also considered.

The future research should include stochastic excitation due to the road roughness and non-stationary curved motion of the vehicle as well as external resonances [5]. Naturally, it is clear that the proposed method may be used for estimation of stability of other mechanical and nonmechanical systems described by differential equations (2.1).

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БЕРТОЛИНО-БАКША СТАБИЛНОСТ ПРИ НЕЛИНЕАРНИМ ВИБРАЦИЈАМА МОТОРНИХ ВОЗИЛА

Резиме. Истраживање одговора возила на неравнине путева је посебно важно са аспекта решавања проблема везаних за динамичку стабилност возила. У овом раду, неравнине путева се сматрају извором нелинеарних вибрација моторних возила. Возило је представљено еквивалентним просторним моделом који има седам степени слободе. Поред решавања симулирањем у оквиру нумеричког кода, спроводи се квази-линеаризација нелинеарних диференцијалних једначина кретања. Решења квази-линеарних диференцијалних једначина принудних вибрација одређена применом методе малог параметра су неопходна за проучавање проблема просторне стабилности возила. Извршена је оптимална стабилизација за упрошћени дводимензионални модел. Укратко се разматрају просторна стабилност и унутрашње резонанце.

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