

CONVERGENCE OF ITERATES WITH ERRORS OF
UNIFORMLY QUASI-LIPSCHITZIAN MAPPINGS
IN CONE METRIC SPACES

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ABSTRACT. The aim of this paper is to consider an Ishikawa type iteration process with errors to approximate the fixed point of two uniformly quasi-Lipschitzian mappings in cone metric spaces. We also extend some fixed point results of these mappings from complete generalized convex metric spaces to cone metric spaces. Our results extend and generalize many known results.

1. INTRODUCTION AND PRELIMINARIES

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method [27], [37], and in optimization theory [8]. K -metric and K -normed spaces were introduced in the mid-20th century (see [27], [37]) by replacing an ordered Banach space instead of the set of real numbers, as the codomain for a metric. L.G. Huang and X. Zhang [11] re-introduced such spaces under the name of cone metric spaces. They and other authors ([1]- [7], [9], [11]-[16], [22]-[24], [26], [28], [32]) proved some fixed point theorems for contractive-type mappings in cone metric spaces.

Consistent with [8] and [11], the following definitions and results will be needed in the sequel.

Key words and phrases. Ishikawa type iteration, uniformly quasi-Lipschitzian mappings, cone metric space, normal and nonnormal cone, fixed point.

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Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

- (a) P is closed, nonempty and $P \neq \{\theta\}$;
- (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
- (c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering \preceq with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P). If $\text{int}P \neq \emptyset$ then P is called a solid cone (see [27]).

There exist two kinds of cones- normal (with the normal constant k) and nonnormal ones [8].

Let E be a real Banach space, $P \subset E$ a cone and \preceq partial ordering defined by P . Then P is called normal if there is a number $k > 0$ such that for all $x, y \in P$,

$$(1.1) \quad \theta \preceq x \preceq y \quad \text{imply} \quad \|x\| \leq k \|y\|,$$

or equivalently, if $(\forall n) x_n \preceq y_n \preceq z_n$ and

$$(1.2) \quad \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = x \quad \text{imply} \quad \lim_{n \rightarrow \infty} y_n = x.$$

The least positive number k satisfying (1.1) is called the normal constant of P . It is clear that $k \geq 1$.

Example 1.1. [27] Let $E = C_{\mathbb{R}}^1[0, 1]$ with $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n \rightarrow \infty} y_n = \theta$, but $\|x_n\| = \max_{t \in [0,1]} \left| \frac{t^n}{n} \right| + \max_{t \in [0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.2) that P is a nonnormal cone.

Definition 1.1. [11], [37] Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies:

- (d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric [11] or K -metric [37] on X and (X, d) is called a cone metric [11] or K -metric space [37] (we shall use the first terms).

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.2.

- (a) [11] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space [11] with normal cone P where $k = 1$.
- (b) For other examples of a cone metric spaces, i.e., P -metric spaces one can see [37], pp. 853-854.

Definition 1.2. [11] Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

- (i) a Cauchy sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;
- (ii) a convergent sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X .

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X .

Let us recall [11] that if P is a normal solid cone, then $x_n \in X$ is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \rightarrow 0$, $n, m \rightarrow \infty$. Further, $x_n \in X$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \rightarrow 0$, $n \rightarrow \infty$.

In the sequel we assume that E is a real Banach space and that P is a normal solid cone in E , that is, normal cone with $intP \neq \emptyset$. The last assumption is necessary in order to obtain reasonable results connected with convergence and continuity. The partial ordering induced by the cone P will be denoted by \preceq .

2. CONVEXITY IN CONE METRIC SPACE

Let (X, d) be a cone metric space with a solid cone P . A mapping $f : X \rightarrow X$ is called asymptotically nonexpansive if there exists $k_n \in [1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$, such that $d(f^n x, f^n y) \preceq k_n d(x, y)$ for all $x, y \in X$. Let $F(f) = \{x \in X : f x = x\}$. If $F(f) \neq \emptyset$, then f is called asymptotically quasi-nonexpansive if there exists $k_n \in [1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$, such that $d(f^n x, p) \preceq k_n d(x, p)$ for all $x \in X, p \in F(f)$. Moreover, it is uniformly quasi-Lipschitzian if there exists $L > 0$ such that $d(f^n x, p) \preceq L d(x, p)$ for all $x \in X, p \in F(f)$.

From the above definition, if $F(f) \neq \emptyset$, it follows that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, and an asymptotically quasi-nonexpansive must be uniformly quasi-Lipschitzian where $L = \sup \{k_n, n \in \mathbb{N}\} < \infty$. In recent years, asymptotically nonexpansive mappings and asymptotically

quasi-nonexpansive mappings have been studied extensively in the setting of convex metric spaces ([10], [17], [18], [19], [20]).

In 1970, Takahashi [25] first introduced a notion of convex metric space which is more general space. It should be pointed out that each linear normed space is a special example of convex metric space, but there exist some convex metric spaces which cannot be embedded into normed space [25].

Now we introduce the following:

Definition 2.1. Let (X, d) be a cone metric space, and $I = [0, 1]$. A mapping $W : X^2 \times I \rightarrow X$ is said to be convex structure on X , if for any $(x, y, \lambda) \in X^2 \times I$ and $u \in X$, the following inequality holds:

$$d(W(x, y, \lambda), u) \preceq \lambda d(x, u) + (1 - \lambda) d(y, u).$$

If (X, d) is a cone metric space with a convex structure W , then (X, d) is called a convex abstract metric space or convex cone metric space (see also [14], [23]). Moreover, a nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$, for all $(x, y, \lambda) \in C^2 \times I$.

Definition 2.2. Let (X, d) be a cone metric space, $I = [0, 1]$, and $\{a_n\}, \{b_n\}, \{c_n\}$ real sequences in $[0, 1]$ with $a_n + b_n + c_n = 1$. A mapping $W : X^3 \times I^3 \rightarrow X$ is said to be convex structure on X , if for any $(x, y, z, a_n, b_n, c_n) \in X^3 \times I^3$ and $u \in X$, the following inequality holds:

$$d(W(x, y, z, a_n, b_n, c_n), u) \preceq a_n d(x, u) + b_n d(y, u) + c_n d(z, u).$$

If (X, d) is a cone metric space with a convex structure W , then (X, d) is called a generalized convex cone metric space. Moreover, a nonempty subset C of X is said to be convex if $W(x, y, z, a_n, b_n, c_n) \in C$, for all $(x, y, z, a_n, b_n, c_n) \in C^3 \times I^3$.

Remark 2.1. If $E = \mathbb{R}$, $P = [0, +\infty)$, $\|\cdot\| = |\cdot|$ then (X, d) is a convex metric space, i.e., generalized convex metric space as in [30].

Example 2.1. Let (X, d) be a cone metric space as in Example 1.2(a). If $W(x, y, \lambda) =: \lambda x + (1 - \lambda)y$, then (X, d) is a convex cone metric space. Hence, this concept is more general than that of a convex metric space.

Definition 2.3. Let (X, d) be a cone metric space with a convex structure $W : X^3 \times I^3 \rightarrow X$, $f, g : X \rightarrow X$ be uniformly quasi-Lipschitzian mappings with

$L > 0$ and $L' > 0$, $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be six sequences in $[0, 1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, n = 1, 2, \dots$. For any given $x_1 \in X$, define a sequence $\{x_n\}$ as follows:

$$(2.1) \quad \begin{cases} y_n &= W(x_n, g^n x_n, v_n, a'_n, b'_n, c'_n), \\ x_{n+1} &= W(x_n, f^n y_n, u_n, a_n, b_n, c_n), \end{cases}$$

where $\{u_n\}, \{v_n\}$ are two sequences in X satisfying the following condition: for any nonnegative integers $n, m, 1 \leq n < m$, if $\delta(A_{nm}) > 0$, then

$$(2.2) \quad \max_{n \leq i, j \leq m} \{\|d(x, y)\| : x \in \{u_i, v_i\}, y \in \{x_j, y_j, f y_j, g x_j, u_j, v_j\}\} < \delta(A_{nm}),$$

where $A_{nm} = \{x_i, y_i, f y_i, g x_i, u_i, v_i : n \leq i \leq m\}, \delta(A_{nm}) = \sup_{x, y \in A_{nm}} \|d(x, y)\|$.

Then $\{x_n\}$ is called the Ishikawa type iteration process with errors of two uniformly quasi-Lipschitzian mappings f and g in convex cone metric space (X, d) .

Remark 2.2. Note that some iteration processes considered in [10], [17], [18], [19] can be obtained from the above process (2.1) as special cases by suitable choosing the spaces and the mappings.

In the sequel, we shall need the following lemma.

Lemma 2.1. [19] *Let nonnegative sequences $\{p_n\}, \{q_n\}, \{r_n\}$ satisfy that*

$$\sum_{n=1}^{\infty} q_n < \infty, \quad \sum_{n=1}^{\infty} r_n < \infty, \quad p_{n+1} \leq (1 + q_n) p_n + r_n, \quad n \geq 1.$$

We have:

- (i) $\lim_{n \rightarrow \infty} p_n$ exists;
- (ii) if $\liminf_{n \rightarrow \infty} p_n = 0$, then $\lim_{n \rightarrow \infty} p_n = 0$.

3. MAIN RESULTS

Now we give our main results of this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete convex cone metric space X , $f, g : C \rightarrow C$ be uniformly quasi-Lipschitzian mappings with $L > 0$ and $L' > 0$, and $F = F(f) \cap F(g) \neq \emptyset$. Suppose that $\{x_n\}$ is the Ishikawa type iteration process with errors defined by (2.1), $\{u_n\}, \{v_n\}$ satisfying (2.2) and let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be six sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + c_n) < \infty.$$

Then, $\{x_n\}$ converges to a common fixed point of f and g if and only if $\liminf_{n \rightarrow \infty} \|d(x_n, F)\| = 0$, where $\|d(x, F)\| = \inf \{\|d(x, q)\| : q \in F\}$.

In order to prove Theorem 3.1 we shall need the following lemma.

Lemma 3.1. *Let C be a nonempty closed convex subset of a complete convex cone metric space X , $f, g : C \rightarrow C$ be uniformly quasi-Lipschitzian mappings with $L > 0$ and $L' > 0$. If the sequence $\{x_n\}$ be as in (2.1) with $\{u_n\}, \{v_n\}$ satisfying (2.2) and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be six sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + c_n) < \infty,$$

and if $F = F(f) \cap F(g) \neq \emptyset$, then:

(i) *there exists a constant vector $v \in P \setminus \{\theta\}$ such that*

$$\|d(x_{n+1}, p)\| \leq k \cdot (1 + \alpha_n L + \alpha_n L L') \cdot \|d(x_n, p)\| + k \cdot \|v\| \alpha_n,$$

for all $n \in \mathbb{N}$ and for all $p \in F$, where k is the normal constant of a cone P ;

(ii) *there exists a real constant $M > 0$ such that*

$$\|d(x_{n+m}, p)\| \leq k \cdot M \cdot \|d(x_n, p)\| + k \cdot M \cdot \|v\| \cdot \sum_{k=n}^{n+m-1} \alpha_k,$$

for all $n, m \in \mathbb{N}$ and for all $p \in F$, where k is the normal constant of a cone P .

Proof. (i) Let $p \in F$ be given. Then we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, f^n y_n, u_n, a_n, b_n, c_n), p) \\ &\preceq a_n d(x_n, p) + b_n d(f^n y_n, p) + c_n d(u_n, p) \\ (3.1) \quad &\preceq a_n d(x_n, p) + b_n L \cdot d(y_n, p) + c_n d(u_n, p), \end{aligned}$$

$$\begin{aligned} d(y_n, p) &= d(W(x_n, g^n x_n, v_n, a'_n, b'_n, c'_n), p) \\ &\preceq a'_n d(x_n, p) + b'_n d(g^n x_n, p) + c'_n d(v_n, p) \\ (3.2) \quad &\preceq a'_n d(x_n, p) + b'_n L' \cdot d(x_n, p) + c'_n d(v_n, p). \end{aligned}$$

Substituting (3.2) into (3.1), it can be obtained that

$$\begin{aligned}
 d(x_{n+1}, p) &\preceq a_n d(x_n, p) + b_n L \cdot (a'_n d(x_n, p) + b'_n L' \cdot d(x_n, p) + c'_n d(v_n, p)) \\
 &\quad + c_n d(u_n, p) \\
 &= (a_n + b_n a'_n L + b_n b'_n L L') d(x_n, p) + b_n c'_n L d(v_n, p) + c_n d(u_n, p) \\
 &\preceq (1 + (1 + L') L b_n) d(x_n, p) + b_n L d(v_n, p) + c_n d(u_n, p) \\
 (3.3) \quad &\preceq (1 + \alpha_n L + \alpha_n L L') d(x_n, p) + \alpha_n v,
 \end{aligned}$$

where $\alpha_n = b_n + c_n, v = Ld(v_n, p) + d(u_n, p)$. Now, (i) follows from (1.1), where k is a normal constant of the cone P .

(ii) It is well known that $1 + x \leq e^x$ for all $x \geq 0$. Using this, from the above inequality (3.3), we have

$$\begin{aligned}
 d(x_{n+m}, p) &\preceq (1 + \alpha_{n+m-1} L + \alpha_{n+m-1} L L') d(x_{n+m-1}, p) + \alpha_{n+m-1} \cdot v \\
 &\preceq e^{\alpha_{n+m-1} L + \alpha_{n+m-1} L L'} [(1 + \alpha_{n+m-2} L + \alpha_{n+m-2} L L') d(x_{n+m-2}, p) \\
 &\quad + \alpha_{n+m-2} \cdot v] + \alpha_{n+m-1} \cdot v \\
 &\preceq e^{L(1+L')(\alpha_{n+m-1} + \alpha_{n+m-2})} d(x_{n+m-2}, p) \\
 &\quad + e^{L(1+L')(\alpha_{n+m-1} + \alpha_{n+m-2})} (\alpha_{n+m-1} + \alpha_{n+m-2}) \cdot v \\
 &\quad \vdots \\
 &\preceq M \cdot d(x_n, p) + \left(M \sum_{k=n}^{n+m-1} \alpha_k \right) \cdot v,
 \end{aligned}$$

where $M = e^{L(1+L') \sum_{k=1}^{+\infty} \alpha_k}$. Further, (ii) follows from (1.1), because P is a normal cone with the normal constant k . □

Proof of Theorem 3.1. Since,

$$\|d(x_n, F)\| = \inf \{ \|d(x_n, q)\| : q \in F \} \leq \inf \{ \|d(x_n, p)\| : \lim x_n = p \in F \} = 0,$$

it follows that the necessity of the conditions is obvious. Thus, we will only prove the sufficiency. From Lemma 3.1(i), we have

$$\|d(x_{n+1}, p)\| \leq k \cdot (1 + \alpha_n L + \alpha_n L L') \cdot \|d(x_n, p)\| + k \cdot \|v\| \cdot \alpha_n.$$

As $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} (a_n + b_n) < \infty$, therefore $\lim_{n \rightarrow \infty} \|d(x_n, F)\|$ exists by Lemma 2.1. According to the hypothesis, $\liminf_{n \rightarrow \infty} \|d(x_n, F)\| = 0$, hence we have that $\lim_{n \rightarrow \infty} \|d(x_n, F)\| = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be a given. There exists an integer n_0 such that for all $n > n_0$ we have

$$\|d(x_n, F)\| < \frac{\varepsilon}{4k^2M} \quad \text{and} \quad \sum_{n=n_0+1}^{\infty} \alpha_n < \frac{\varepsilon}{4k^2\|v\|M}.$$

In particular, there exists a $p_1 \in F$ and an integer $n_1 > n_0$ such that

$$\|d(x_{n_1}, p_1)\| < \frac{\varepsilon}{4k^2M}.$$

It follows from Lemma 3.1(ii), that when $n > n_1$, we get

$$\begin{aligned} \|d(x_{n+m}, p_1)\| &= \|d(x_{n_1+(n+m-n_1)}, p_1)\| \\ (3.4) \quad &\leq k \cdot M \cdot \|d(x_{n_1}, p_1)\| + k \cdot M \cdot \|v\| \cdot \sum_{k=n_1}^{n+m-1} \alpha_k \end{aligned}$$

and

$$\begin{aligned} \|d(x_n, p_1)\| &= \|d(x_{n_1+(n-n_1)}, p_1)\| \\ (3.5) \quad &\leq k \cdot M \cdot \|d(x_{n_1}, p_1)\| + k \cdot M \cdot \|v\| \cdot \sum_{k=n_1}^{n-1} \alpha_k. \end{aligned}$$

Therefore, from (1.1), (3.4) and (3.5), we obtain that

$$\begin{aligned} \|d(x_{n+m}, x_n)\| &\leq k \cdot \|d(x_{n+m}, p_1) + d(p_1, x_n)\| \\ &\leq k \cdot \|d(x_{n+m}, p_1)\| + k \cdot \|d(p_1, x_n)\| \\ &\leq 2k^2 \cdot M \cdot \|d(x_{n_1}, p_1)\| + k^2 \cdot \|v\| \cdot M \cdot \left(\sum_{k=n_1}^{n+m-1} \alpha_k + \sum_{k=n_1}^{n-1} \alpha_k \right) \\ &\leq 2k^2 \cdot M \cdot \|d(x_{n_1}, p_1)\| + 2k^2 \cdot \|v\| \cdot M \cdot \sum_{k=n_1}^{n+m-1} \alpha_k \\ &< 2k^2 \cdot M \cdot \frac{\varepsilon}{4k^2M} + 2k^2 \cdot \|v\| \cdot M \cdot \frac{\varepsilon}{4k^2\|v\|M} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in closed convex subset of a complete cone metric space. Therefore, it must converge to a point of C . Suppose $\lim_{n \rightarrow \infty} x_n = p$. We will prove that $p \in F$.

For given $\varepsilon > 0$, there exists an integer n_2 such that for all $n \geq n_2$, we have

$$(3.6) \quad \|d(x_n, p)\| < \frac{\varepsilon}{2k(1+L+L')} \quad \text{and} \quad \|d(x_n, F)\| < \frac{\varepsilon}{2k(1+3(L+L'))}.$$

In particular, there exists a $p_2 \in F$ and integer $n_3 > n_2$ such that

$$(3.7) \quad \|d(x_{n_3}, p_2)\| < \frac{\varepsilon}{2k(1+3(L+L'))}.$$

First, we have

$$\begin{aligned} d(fp, p) &\preceq d(fp, p_2) + d(p_2, x_{n_3}) + d(x_{n_3}, p) + 2d(fx_{n_3}, p_2) \\ &\preceq Ld(p, p_2) + d(x_{n_3}, p_2) + d(x_{n_3}, p) + 2Ld(x_{n_3}, p_2) \\ &\preceq Ld(p, x_{n_3}) + Ld(x_{n_3}, p_2) + d(x_{n_3}, p_2) + d(x_{n_3}, p) + 2Ld(x_{n_3}, p_2) \\ &= (1 + L)d(x_{n_3}, p) + (1 + 3L)d(x_{n_3}, p_2). \end{aligned}$$

Now, according to (1.1), (3.6) and (3.7) we obtain

$$\begin{aligned} \|d(fp, p)\| &\leq k(1 + L)\|d(x_{n_3}, p)\| + k(1 + 3L)\|d(x_{n_3}, p_2)\| \\ &< k(1 + L)\frac{\varepsilon}{2k(1 + L + L')} + k(1 + 3L)\frac{\varepsilon}{2k(1 + 3(L + L'))} < \varepsilon, \end{aligned}$$

and we also have $\|d(gp, p)\| < \varepsilon$. Since ε is arbitrary, it follows that $d(fp, p) = d(gp, p) = \theta$, that is, p is a fixed point of f and g . This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete convex cone metric space X , $f, g : C \rightarrow C$ be asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$ and $\{k'_n\}$ (without the conditions $\sum_{n=1}^{+\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{+\infty} (k'_n - 1) < \infty$), and $F = F(f) \cap F(g) \neq \emptyset$. Suppose that $\{x_n\}$ is the Ishikawa type iteration process with errors defined by (2.1), $\{u_n\}, \{v_n\}$ satisfy (2.2) and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are six sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + c_n) < \infty.$$

Then $\{x_n\}$ converges to a common fixed point of f and g if and only if $\liminf_{n \rightarrow \infty} \|d(x_n, F)\| = 0$, where $\|d(x, F)\| = \inf \{\|d(x, q)\| : q \in F\}$.

Proof. Since $\{k_n\}, \{k'_n\} \subset [1, +\infty)$ and $\lim_{n \rightarrow \infty} k_n = \lim_{n \rightarrow \infty} k'_n = 1$, then there exist $L > 0$ and $L' > 0$ such that $L = \sup_{n \geq 0} \{k_n\} < \infty$ and $L' = \sup_{n \geq 0} \{k'_n\} < \infty$. In this case, f, g are uniformly quasi-Lipschitzian mappings with $L > 0$ and $L' > 0$. Hence, Theorem 3.2 can be proven by Theorem 3.1. □

Theorem 3.3. *Let K be a nonempty closed convex subset of a complete convex cone metric space X , $f, g : C \rightarrow C$ be asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\{k'_n\}$ (without the conditions $\sum_{n=1}^{+\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{+\infty} (k'_n - 1) < \infty$), and $F = F(f) \cap F(g) \neq \emptyset$. Suppose that $\{x_n\}$ is the Ishikawa type iteration process*

with errors defined by (2.1), $\{u_n\}$, $\{v_n\}$ satisfy (2.2) and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are six sequences in $[0, 1]$ satisfying

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + c_n) < \infty.$$

Then $\{x_n\}$ converges to a common fixed point of f and g if and only if $\liminf_{n \rightarrow \infty} \|d(x_n, F)\| = 0$, where $\|d(x, F)\| = \inf \{\|d(x, q)\| : q \in F\}$.

Proof. It is clear that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. Therefore, Theorem 3.3 can be proven by Theorem 3.1. \square

Corollary 3.1. *In Theorem 3.1 by setting $E = \mathbb{R}$, $P = [0, \infty)$, $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$ (that is $\|\cdot\| = |\cdot|$), we get Lemma 2 and Theorems 1, 2, 3 from [29].*

Corollary 3.2. *When in Theorem 3.1 we set $E = \mathbb{R}$, $P = [0, \infty)$, $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$ (that is $\|\cdot\| = |\cdot|$), we obtain the main result from [7], Theorems 2.1 and 2.2 from [30] and Theorem 2.1 and Corollary 2.3 from [31].*

Remark 3.1. It is worth noticing that according to our Theorem 3.1 the condition $F = F(S) \cap F(T) \neq \emptyset$ in the Theorem 2.1 and Corollary 2.3 in [30] and in the Theorems 2.1 and 2.2 and Corollaries 2.1 and 2.2 in [31] is superfluous. Indeed, it is very easy to prove that in both cases ([30] page 1 and [31] page 1, (*)) the set F is singleton and S, T are two uniformly quasi-Lipschitzian mappings (see also [2]).

Remark 3.2. We believe that the results of our paper can be extended in the frame of the cone pseudometrics, cone uniform and cone locally convex spaces (for details see [33]-[36]). Also, all main results from [21] can be extended in the frame of convex cone metric spaces as in our paper.

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REFERENCES

- [1] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420.
- [2] M. Abbas, Mirko Jovanović, S. Radenović, Aleksandra Sretenović, Suzana Simić, *Abstract metric spaces and approximating fixed points of a pair of contractive type mappings*, Journal of Computational Analysis and Applications, vol. **13**, no. 2, (2011), 243–253.
- [3] I. Altun, and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circolo Mat. Palermo, **58** (2009), 319–325.

- [4] I. Altun, V. Rakočević, *Ordered cone metric spaces and fixed point results*, *Comput. Math. Appl.* **60** (2010), 1145–1151.
- [5] I. D. Arandjelović, D. J. Kečkić, *A counterexample on a theorem of Khojasteh, Goodarzi and Razani*, *Fixed Point Theory Appl.* Vol. 2010, Article ID 470141, 6 pages, doi: 10.1155/2010/470141.
- [6] C. Di Bari, and P. Vetro, φ -pairs and common fixed points in cone metric spaces, *Rend. Circolo Mat. Palermo*, **57** (2008), 279–285.
- [7] S. C. Bose, *Common fixed points of mappings in a uniformly convex Banach space*, *Journal of the London Mathematical Society*, **18** (1978), no. 1, 151–156.
- [8] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.
- [9] Lj. Gajić, D. Ilić, V. Rakočević, *On Ćirić maps with a generalized contractive iterate at a point and Fisher's quasi-contractions in cone metric spaces*, *Applied Mathematics and Computation* **216** (8) (2010), 2240–2247.
- [10] Hafiz Fukhar-ud-din, Safer Hussain Khan, *Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications*, *J. Math. Anal. Appl.* **328** (2007), 821–829.
- [11] L. G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, *J. Math. Anal. Appl.* **332** (2) (2007), 1468–1476.
- [12] D. Ilić, V. Rakočević, *Common fixed points for maps on cone metric space*, *J. Math. Anal. Appl.* **341** (2008), 876–882.
- [13] D. Ilić, V. Rakočević, *Quasi-contraction on cone metric space*, *Applied Mathematics Letters* **22** (2009), 728–731.
- [14] S. Janković, Z. Kadelburg, S. Radenović and B. E. Rhoades, *Assad-Fixed-Type Point Theorems for a Pair of Nonself Mappings on Cone Metric Spaces, Fixed Point Theory and Applications*, Volume 2009, Article ID 761086, 16 pages, doi:10.1155/2009/761086.
- [15] S. Janković, Z. Kadelburg, and S. Radenović, *On cone metric spaces: A survey*, *Nonlinear Analysis*, **74** (2010), 2591–2601.
- [16] Z. Kadelburg, S. Radenović, and V. Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, *Applied Mathematics Letters* **24** (2011), 370–374.
- [17] Jae Ug Jeong, Soo Hwan Kim, *Weak and strong convergence of the Ishikawa iteration process with errors for two asymptotically nonexpansive mappings*, *Applied Mathematics and Computation* **181** (2006), 1394–1401.
- [18] You-Xian Tian, *Convergence of an Ishikawa Type Iterative Scheme for Asymptotically Quasi-Nonexpansive Mappings*, *Computers and Mathematics with Applications* **49** (2005), 1905–1912.
- [19] Liu Qihou, *Iterative Sequences for Asymptotically Quasi-nonexpansive Mappings with errors member*, *J. Math. Anal. Appl.* **259** (2001), 18–24.
- [20] Liu Qihou, *Iterative Sequences for Asymptotically Quasi-nonexpansive Mappings*, *J. Math. Anal. Appl.* **259** (2001), 1–7.
- [21] Qing-you Liu, Zhi-bin Liu, Nan-jing Huang, *Approximating the Common Fixed Points of Two Sequences of Uniformly Quasi-Lipschitzian Mappings in Convex Metric Spaces*, *Applied Mathematics and Computation* **216** (2010), 883–889.
- [22] S. Radenović, *Common fixed points under contractive conditions in cone metric spaces*, *Computers and Mathematics with Applications* **58** (2009), 1273–1278.
- [23] S. Radenović, B. E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, *Computers and Mathematics with Applications*, **57** (2009), 1701–1707.
- [24] S. Radojević, Lj. Paunović, S. Radenović, *Abstract metric spaces and Hardy-Rogers-type theorems*, *Applied Mathematics Letters* **24** (2010), 553–558.
- [25] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, *Kodai. Math. Sem. Rep.* **22** (1970), 142–149.
- [26] D. Turkoglu and M. Abuloha, *Cone metric Spaces and Fixed Theorems in Diametrically Contractive Mappings*, *Acta Mathematica Sinica, English Series Mar.*, 2010, Vol. **26**, No 3, 489–496.

- [27] J. Vandergraft, *Newton method for convex operators in partially ordered spaces*, SIAM J. Numer. Anal. **4** (1967), 406–432.
- [28] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circolo Math. Palermo, **56** (2007), 464–468.
- [29] Chao Wang, Li-wei Liu, *Convergence theorems for fixed points of uniformly quasi-Lipschitzian mappings in convex metric spaces*, Nonlinear Analysis **70** (2009), 2067–2071.
- [30] Chao Wang, Jinghao Zhu, Boško Damjanović, Liang-gen Hu, *Approximating fixed points of a pair of contractive type mappings in generalized convex metric spaces*, Applied Mathematics and Computation **215** (2009), 1522–1525.
- [31] Chao Wang, Jin Li, and Daoli Zhu, *Convergence Theorems for the Unique ommon Fixed Point of a Pair of Asymptotically Nonexpansive Mappings in Generalized Metric Spaces*, Fixed Point Theory and Applications, Volume 2010, Article ID 281890, 6 pages, doi: 10.1155/2010/281890.
- [32] D. Wardowski, *Endpoints and fixed points of set-valued contractions in cone metric spaces*, Nonlinear Analysis **71** (2010), 512–516.
- [33] K. Włodarczyk, R. Plebaniak, M. Dolinski, *Cone uniform, cone locally convex and cone metric spaces, endpoints, set-valued dynamic system and quasi-asymptotic contractions*, Nonlinear Analysis **71** (2009), 5022–5031.
- [34] K. Włodarczyk, R. Plebaniak, C. Obczynski, *Convergence theorems, best approximations and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces*, Nonlinear Analysis **72** (2010), 794–805.
- [35] K. Włodarczyk, R. Plebaniak, *Periodic Point, Endpoint, and Convergence Theorems for Dissipative Set-Valued Dynamic Systems with Generalized Pseudodistances in Cone Uniform and Uniform Spaces*, Fixed Point Theory and Applications, Volume 2010, Article ID 864536, 32 pages, doi: 10.1155/2010/864536.
- [36] K. Włodarczyk, R. Plebaniak, *Maximality principle and general results of Ekeland and Caristi types without lower semicontinuity assumptions in cone uniform spaces with generalized pseudodistances*, Fixed Point Theory and Applications, Volume 2010, Article ID 175453, 35 pages, doi:10.1155/2010/175453.
- [37] P. P. Zabrejko, *K-metric and K-normed spaces: survey*, Collect. Math **48** (1997), 852–859.

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