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# EXPERIMENTAL AND NUMERICAL APPROACH TO NATURAL FREQUENCY OF TAPERED 3D PRINTED CANTILEVER BEAM A TIP BODY

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**Abstract**: This paper presents the continuation of the research by the authors on the topics of vibration analysis. Namely, a tapered cantilever beam has been studied in theoretical and experimental terms. The beam is 3D printed using the FDM technique of PLA material with two characteristics patterns, linear layered and the two-dimensional periodic structure. The former of the two is conducted to tensile test in order to determine elasticity modulus, so the computations of natural frequencies could be conducted. The experimental part of the research, in terms of natural frequency, is conducted on six specimens with various eccentricities. The angular frequencies of one representative sample, obtained by analytical computations and experimental measurement are compared.

**Key-words**: coupled vibrations, natural frequency, 3D printed material, cantilever beam

## 1. Introduction

Elastic beams have been widely used and studied in the structural mechanics. Due to the increasing use of additive manufacturing processes, novel materials have become more available for structural design. Also, functionally graded materials have been present for almost half a century [1]. The use of 3D printed and functionally graded materials have several advantages. In order to tackle vibration problem of inhomogeneous structures and compute natural frequencies several procedures have been developed. One of them is the Symbolic Numeric Method of Initial Parameters in Differential Form (SNMIP) which enables researchers to solve natural frequencies directly from governing equations [2] for inhomogeneous beams with variable cross sectional characteristics. Also, there are several methods based on discretization of elastic elements as presented by the researcher in [3].

The 3D printed material is inhomogeneous and its mass and mechanical characteristics may differ from the homogeneous material of the used filament. Mechanical parameters of the 3D printed material have been analyzed in [4] and the impact of the inclination angle among layers on the mechanical properties has been discussed.

Material design is an emerging engineering and scientific field. Many authors look for inspiration in natural shapes or traditional patterns in order to find optimal lattice shape for vibration control or energy harvesting purposes as in [5-7].

In this paper the material for cantilever beam is produced by 3D printing and two characteristic patterns are be presented, the linear-layered and the two dimensional periodic structure. For both pattern types, the fundamental natural frequency of cantilever beams is measured. Furthermore, for one case the comparison with computed results is conducted. For linear layered structure, the mass and mechanical characteristics have been determined in order to examine the deviation of characteristics of printed material to homogeneous filament. Measured results were used for computation of natural frequencies.

## 2. Formulation of the problem

The natural frequencies of an elastic tapered cantilever beam with the rigid body eccentrically displaced with respect to the free end are investigated. The eccentricity causes coupling between axial and bending vibrations. Similar problem was theoretically analyzed in the paper [8] for an axially functionally graded cantilever beam. In this paper the coupled vibrations of a 3D printed cantilever beam will be examined by practical measurement. The rigid body at the free end of the beam will be introduced via accelerometer used for measuring. Three cases of mass eccentricities and two different patterns in material design will be used.

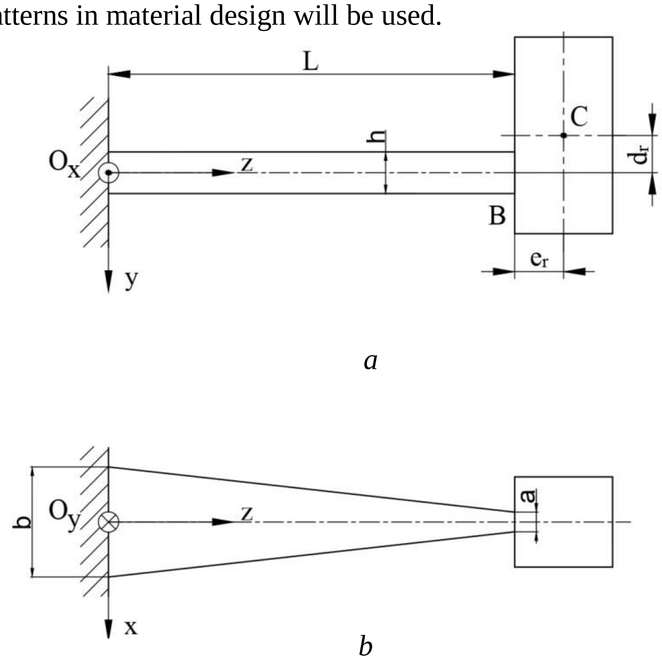



Fig. 1. An undeformed shape of the tapered cantilever beam with a tip mass presented schematically in: *a* – front and *b* – top view

Governing equations of motion in transverse and axial direction of presented cantilever beam, under the assuption of validity of Euler-Bernouli beam theory, read as [9]

$$\frac{\partial}{\partial z}\left[F_t(z,t)\right]-\rho(z)A(z)\frac{\partial^2 w(z,t)}{\partial t^2}=0,\frac{\partial}{\partial z}\left[F_a(z,t)\right]-\rho(z)A(z)\frac{\partial^2 u(z,t)}{\partial t^2}=0 \quad (1.1)$$

Even though the equations are not mutually coupled, the coupling originates from boundary conditions as it will be presented.

After implementing the method of separation of variables as presented in [9], one may write a system of ordinary differential equations in matrix form as in [8]

$$\frac{d}{dz}\boldsymbol{X}(z)=\boldsymbol{T}(z)\times\boldsymbol{X}(z) \quad (1.2)$$

where a vector ***X(z)*** and a vector of derivatives of arguments of ***X(z)*** and a matrix ***T(z)*** are as in [8]

$$\boldsymbol{X}(z)=\begin{bmatrix}U(z) & W(z) & W'(z) & F_a(z) & F_t(z) & M_f(z)\end{bmatrix}^T \quad (1.3)$$

$$\frac{d}{dz}\boldsymbol{X}(z)=\begin{bmatrix}\frac{dU(z)}{dz} & \frac{dW(z)}{dz} & \frac{dW'(z)}{dz} & \frac{dF_a(z)}{dz} & \frac{dF_t(z)}{dz} & \frac{dM_f(z)}{dz}\end{bmatrix}^T \quad (1.4)$$

$$\boldsymbol{T}(z)=\begin{bmatrix}0 & 0 & 0 & \frac{1}{E(z)A(z)} & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{E(z)I_x(z)}\\ -\omega^2\rho(z)A(z) & 0 & 0 & 0 & 0 & 0\\ 0 & -\omega^2\rho(z)A(z) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \quad (1.5)$$

The free body diagram is presented in Fig. 2. and it creates the foundation for the derivation of boundary conditions.

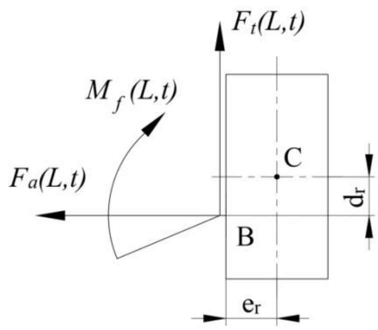



Fig. 2. A free body diagram of the attached rigid body

Boundary conditions at the left and right end of the cantilever beam (*z=0, z=L*) shown in Figure 1 read as in [8]

$$U(0)=0, W(0)=0, W'(0)=0 \quad (1.6)$$

$$-\omega^2 J_{C_r}W'(L)=M_f(L)+F_a(L)d_r+F_t(L)e_r, \quad (1.7)$$

$$m_r\omega^2\left[U(L)+W'(L)d_r\right]=F_a(L),m_r\omega^2\left[W(L)+W'(L)e_r\right]=F_t(L) \tag{1.8}$$

The third, rotational motion equation is written about mass center *C*, thus $J_{Cr}$ stands for the centroidal mass moment of inertia for *x* axis.

## 3. Numerical procedure

By using the Symbolic numeric procedure in differential form, as presented in [2,10] the angular frequencies will be determined. Similarly to the model presented in [8], based on the linearity of the presented system, and the implementation of the method of initial parameters in differential form one may write a solution, as a sum of particular solutions as

$$\boldsymbol{X}(z,\omega)=C_1\boldsymbol{X_1}(z,\omega)+C_2\boldsymbol{X_2}(z,\omega)+C_3\boldsymbol{X_3}(z,\omega) \tag{1.9}$$

Where, $C_1$, $C_2$ and $C_3$ are integration constants. Particular solutions, with respect to axial coordinate and angular frequency as a symbolic variables in vector form may read:

$$\boldsymbol{X_i}(z,\omega)=\begin{bmatrix}U_i(z,\omega) & W_i(z,\omega) & W_i^{'}(z,\omega) & F_{ai}(z,\omega) & F_{ti}(z,\omega) & M_{fi}(z,\omega)\end{bmatrix}^T, i=1,2,3 \tag{1.10}$$

Presented particular solutions shall satisfy the folowing boundary conditions (z=0)

$$\boldsymbol{X}_1(0,\omega)=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^T, \boldsymbol{X}_2(0,\omega)=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}^T, \\ \boldsymbol{X}_3(0,\omega)=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}^T \tag{1.11}$$

According to the SNMIP procedure, let us define expressions $a_{ji}(\omega)$, $i,j$=1,2,3 based on the boundary conditions at the free end of the beam

$$a_{1i}(\omega)=-F_{ai}(L,\omega)+m_r\omega^2\left(U_i(L,\omega)+W_i^{'}(L,\omega)d_r\right), i=1,2,3 \tag{1.12}$$

$$a_{2i}(\omega)=-F_{ti}(L,\omega)+m_r\omega^2\left(W_i(L,\omega)+W_i^{'}(L,\omega)e_r\right), i=1,2,3 \tag{1.13}$$

$$a_{3i}(\omega)=M_{fi}(L,\omega)+\omega^2 J_{Cr}W_i^{'}(\mathrm{L},\omega)+F_a(\mathrm{L},\omega)d_r+F_t(\mathrm{L},\omega)e_r, i=1,2,3 \tag{1.14}$$

The -system of equations, obtained in that manner one may read

$$\begin{bmatrix}a_{11}(\omega) & a_{12}(\omega) & a_{13}(\omega)\\ a_{21}(\omega) & a_{22}(\omega) & a_{23}(\omega)\\ a_{31}(\omega) & a_{32}(\omega) & a_{33}(\omega)\end{bmatrix}\begin{bmatrix}C_1\\ C_2\\ C_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \tag{1.15}$$

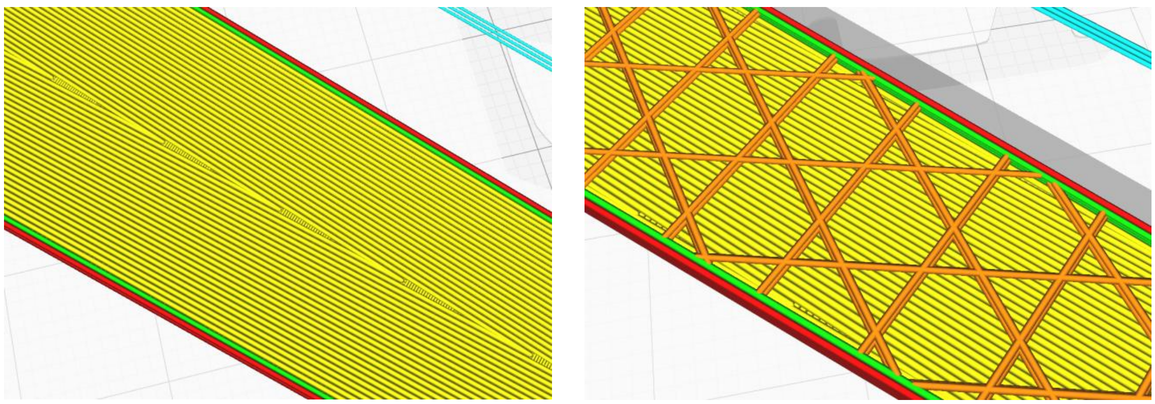
The latter system of equations will have nontrivial solutions if

$$h(\omega)\equiv\begin{vmatrix}a_{11}(\omega) & a_{12}(\omega) & a_{13}(\omega)\\ a_{21}(\omega) & a_{22}(\omega) & a_{23}(\omega)\\ a_{31}(\omega) & a_{32}(\omega) & a_{33}(\omega)\end{vmatrix}=0 \tag{1.16}$$

Where $h(\omega)$ presents the frequency equation of the presented problem from which the angular frequency can be determined for coupled axial and bending vibrations.

## 4. Material structure

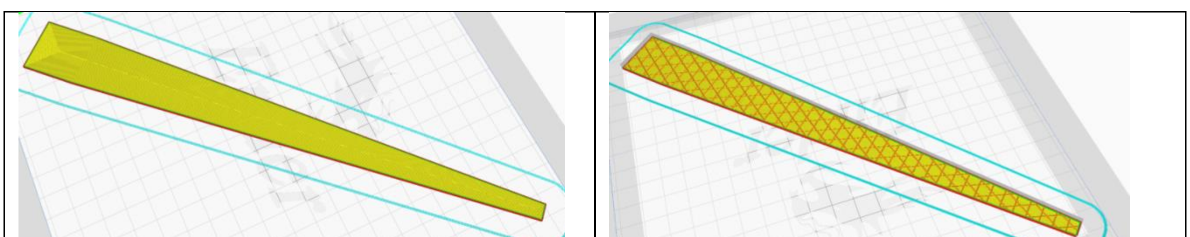
The Fuse Deposition Modeling (FDM) technique is applied for printing a cantilever beam of Polylactic Acid (PLA) material. The reason for choosing the PLA as the material for this study is in its mechanical and thermal characteristics. These characteristics enable the detailed printing that can be exploited when printing periodic structures. Thus, two representative structures are presented in Fig. 3. The linear layered structure is presented in Fig. 3a, while the two-dimensional periodic structure is presented in Fig. 3b.



*a* *b*

Fig. 3. 3D printed material structure: *a* – linear layered material, *b* – two-dimensional periodic structure

The two cantilever beams of same dimensions will be introduced with different internal structures. One will have linear layered structure and the second one will have a two-dimensional periodic structure as presented in Fig. 3. The comparable dimensions with different inner structure will result in different frequencies. In Fig. 4a, the cantilever beam of linear layered structure prepared for 3D printing is presented while Fig. 4b shows the beam of two-dimensional periodic structure.



*a* *b*

Fig. 4. Beams prepared for 3D print: *a* – linear layered structured beam, *b* – two-dimensional periodic structured beam

## 5. Mechanical testing

### 5.1 Preparation for mechanical testing

The experimental determination of elasticity modulus is conducted only for linear layered structure using specimens that are designed according to ISO 527-2-2012 (International standard, plastics determination of tensile properties Part 2: Test conditions for molding and extrusion plastics) as in [4]. The zoomed structure is presented in Fig. 5, and six specimens prepared for testing are presented in Fig. 6, while characteristic dimensions are presented in Table 1.

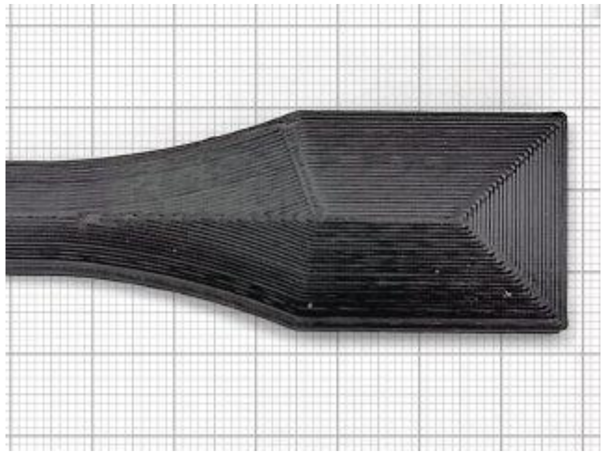


Fig. 5. 3D printed linear layered structure

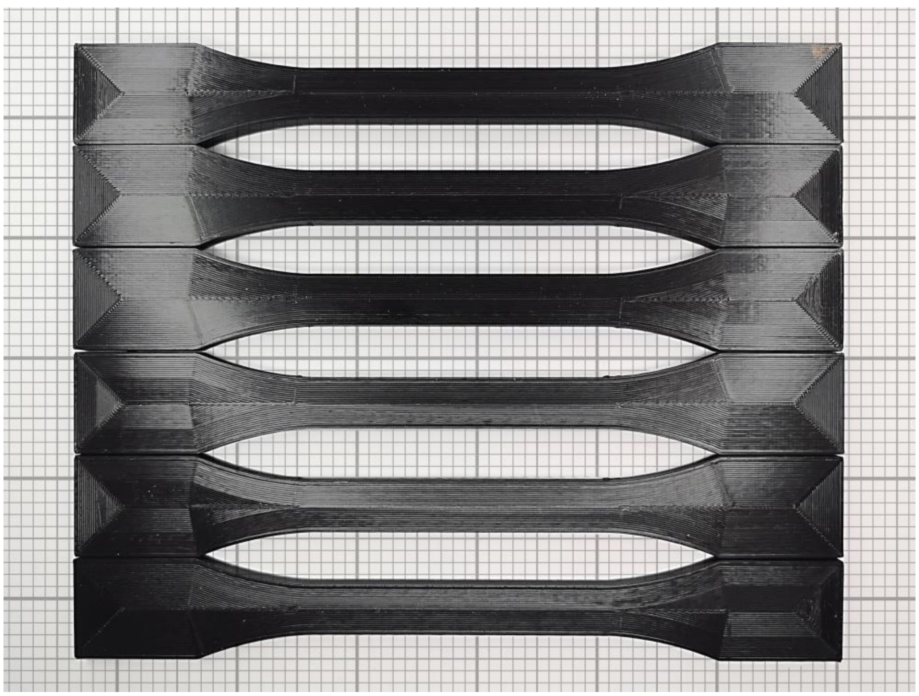


Fig. 6. Testing specimens

| | |
|---|---|
| Thickness, mm | 4 |
| Width, mm | 10.3 |
| Gauge Length, mm | 110 |

Table 1. Dimensions of tested specimens

### 5.2 Elasticity modulus

In order to examine the elasticity modulus, the universal tensile test machine, see Fig. 7, is used. Testing is conducted under standard conditions on six specimens and tested data is collected using digital collector. In Table 2 experimental results are presented.

During tensile testing eight specimens are used, of which two were exploited for setting up test machine and data acquisition, while six are considered to be relevant test samples.

| Sample number | Elasticity modulus, MPa |
|---|---|
| 1 | 2334.37 |
| 2 | 2816.3 |
| 3 | 2774.7 |
| 4 | 2800.04 |
| 5 | 2732.44 |
| 6 | 2763.4 |

Table 2. Test results of six samples

Average result is 2703.54 MPa with standard deviation of 183.192 MPa.

Fig. 7. Tensile test machine

Mass density and average elasticity modulus of printed material are presented in Table 3. It can be noted that the mass and mechanical characteristics of printed element decrease when compared to the homogeneous PLA material. When compared to 3D printed material in the paper [4] slight deviations can be observed.

| Mass density (kg/m$^3$) | Elasticity modulus (MPa) |
|---|---|
| 1.170,35 | 2703.54 |

Table 3. Average characteristics of the material of cantilever beam

The mechanical properties of linear layered structure will be used for the computations of angular frequencies of cantilever beam.

## 6. Determination of frequencies

The fundamental natural frequency of a cantilever beam has been determined experimentally for six configurations. One configuration has been chosen as the representative and angular frequencies have been computed numerically.

### *6.1 Experimental approach*

The structure prepared for experiment consists of: the cantilever beam prepared for experiment, the 3D printed adaptive ring for accelerometer placement and the accelerometer itself, as modeled in Fig. 8. The accelerometer presents the rigid body eccentrically displaced with respect to the free tip of the cantilever beam.

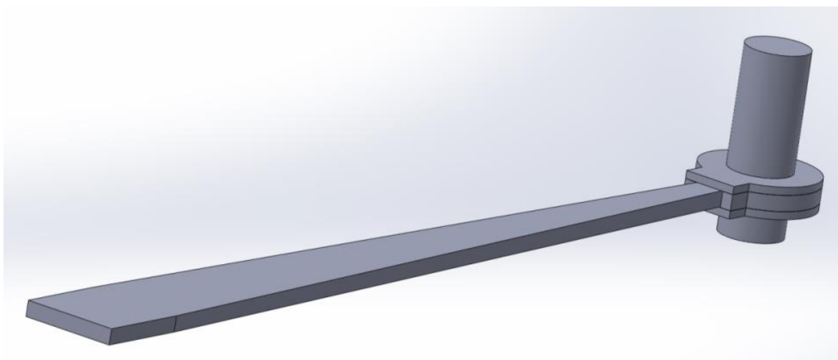


Fig. 8. Schematic representation of tapered cantilever beam with eccentrically displaced rigid body

The following equipment is applied for the measurement of the natural frequency: Microlog CMVA60-SKF with Wilcoxon 793 accelerometer. The accelerometer is a high performance general purpose with characteristics 100mV/g.

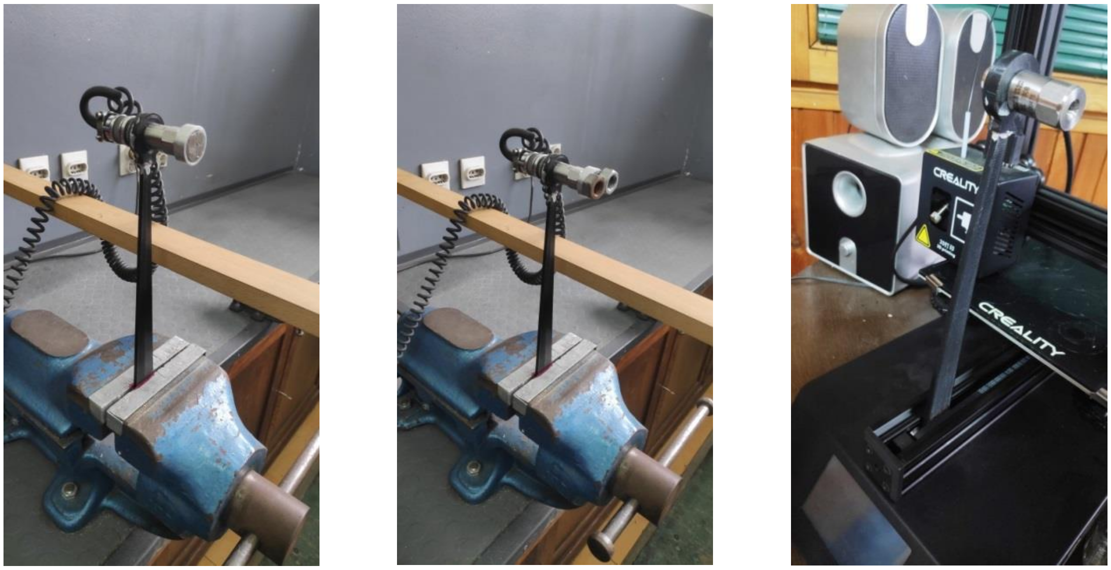
In order to obtain different eccentricities different masses were placed at the end of the beam.

By using the FFT analysis the results for the obtained natural frequency are presented in Table 4.

| Measurement | f [Hz] | ω [rad/s] |
|---|---|---|
| 1a | 3.375 | 21.195 |
| 1b | 3 | 18.84 |
| 1c | 3.687 | 23.15436 |
| 2a | 3.562 | 22.36936 |
| 2b | 3.125 | 19.625 |
| 2c | 3.937 | 24.72436 |

Table 4. Measured Natural frequencies/Angular frequencies

Different experimental setups are presented in Fig. 9., the number denotes the type of material structure and the letter mass of the eccentrically placed body. Number 1 stands for two-dimensional pattern, while, 2 stands for linear layered structure. Letter “a” stands for the case where rigid body consists of accelerometer and additional weight, “b” stands for case “a” with additional two M12 screws and “c” shows accelerometer without attached masses.




Configuration 1a and 2a | Configuration 1b and 2b | Configuration 1c and 2c

*a* | *b* | *c*

Fig. 9. Experimental setup

*6.2 Numerical approach*

A representative sample is considered. The fundamental natural frequency is computed using the Symbolic numeric method of initial parameters in differential form as explained in Chapter 3 of the paper.

Geometrical characteristics of the cross section with respect to the axial coordinate (*z*) are:

$$A(z) = 0.004(0.025 - 0.0625z); I_x(z) = (0.025 - 0.0625z)(0.004 \wedge 3)/12. \tag{1.17}$$

The free length of the cantilever beam is L=228mm.

| Elasticity modulus (MPa) | | |
|---|---|---|
| 2520.34 | 2703.54 | 2886.7 |
| $\omega$ [rad/s] | | |
| 24.667384 | 25.548178 | 26.399423 |

Table 5. Computed natural frequencies for configuration 2c

**5. Conclusions**

A detailed study on coupled bending and axial vibrations of 3D printed cantilever beams with different boundary conditions is conducted in this paper. The study presents an upgrade of the practical work of the group of authors that studied coupled vibrations of axially functionally graded on theoretical level. The two material structures are designed, namely a linear layered and a two-dimensional periodic one. For each of the structures, the three different boundary conditions were imposed by changing mass and mechanical characteristics of the rigid body at the free end. The linear layered structured material is conducted to tensile test in order to obtain the elasticity modulus, which is implemented in numerical calculations. One experimental case has been chosen as the representative, and computations have been conducted. The obtained results of the measured and computed angular frequencies are comparable. Cantilever beams of specific material design are treated in this paper, both experimentally and numerically. Numerical model is applicable to the general beam designed, yet limitations are in the available experimental capacity. The presented paper is the introduction to more complex research in the field of vibrations of periodic structures made of novel materials.

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