Nonlocal axial vibration of a fractional order viscoelastic nanorod

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Summary. This communication presents a novel nonlocal fractional order viscoelastic model of nanorod. The main assumptions for the proposed model are discussed. Solution of the motion equation involving fractional order derivative is presented. Influences of a nonlocal parameter and fractional order derivative on the free damped vibration of nanorod are presented through numerical examples.

Introduction

Nanostructure materials are widely examined by scientists due to the rising need for application in nanosensors, nanoactuators, nanoopto-mechanical systems, nanoelectro-mechanical devices etc. Such based new materials offer considerable benefits compared to conventional materials. The nonlocal theory of Eringen shown to be very efficient in modelling of nanostructures and incorporates the atomic forces and internal length scale into a model [1].

Nanorods can be grown using various methods within the range from 1 to 100 nanometers. Several types of grown materials can be considered to have a nanorod structure. Here we mention only carbon nanorods (multi-wall carbon nanotubes), boron nitride and zinc oxide nanorods. In such small systems it is necessary to take into account small-scale (nonlocal) effects. Vibration analysis of nanorods is significant for the application of these systems in engineering practice. Therefore, the nonlocal theory and classical methods used in vibration analysis of macro structures can be employed in order to investigate the vibration behavior of nanorods. However, nonlocal elasticity models of nanorods do not take in to account structural damping, thus nonlocal viscoelastic constitutive relations need to be considered [2]. There are many linear viscoelastic models available in the literature such as Kelvin-Voight, Maxwel and Standard Linear Solid. These models are originally given with integer order derivatives and in some cases, too many physical parameters are needed to fit experimental results. In order to reduce the number of parameters one may introduce modified linear viscoelastic models involving fractional order derivatives. Such models can pronounce more elastic or viscous behavior by changing the order of fractional derivative.

There are many definitions of fractional derivatives available in the literature [3]. The Riemann-Liouville definition is used very often in viscoelastic models of structures [4]. Deficiencies of this definition such as fractional order initial conditions with questionable physicality can be avoid under certain assumptions. Solutions of fractional order differential equations can be find using analytical methods or numerical approximation methods depending of the problem under consideration [5].

In this work, we analyzed the free vibration of a nonlocal viscoelastic nanorod. Equation of motion of the system which includes the small-scale effects is derived using D' Alembert's principle with nonlocal Kelvin-Voight constitutive relation involving fractional order derivatives. The system is analyzed for clamped-free (C-F) boundary conditions. Solution of partial differential equation is proposed in the form of separation of variables. Obtained fractional order differential equation in terms of a time function is solved using the Laplace transform method. Solution of the equation in the Laplace domain is expanded into a convergent series and using inverse Laplace transform presented into the time domain.

Constitutive equation of a nonlocal fractional order viscoelastic model

Using the operator D^{γ} for the left Riemann-Liouville derivative [2] we can write the modified Kelvin-Voigt constitutive relation for uniaxial deformation of one dimensional structure which takes into account nonlocal effects and involves fractional order derivative of a strain in the form

$$\sigma - \mu \frac{d^2\sigma}{dx^2} = E_0\left[\varepsilon + \tau_0^{\gamma} D^{\gamma}\left(\varepsilon\right)\right] \tag{1}$$

where $\mu = \left(e_0 a\right)^2$ is nonlocal parameter, a denotes internal characteristic length, e_0 is material constant, σ and ε are nonlocal stress and strain , E_0 and τ_0 are elastic modulus and relaxation time, respectively.

Motion equation

Let as consider a C-F nanorod of a length L and constant cross sectional area A along the x coordinate. Material of the nanorod is homogenous. We assume the free longitudinal vibration of a nanorod in x direction. Force N is resultant of an axial stress σ acting internally on A and p is the axially distributed force which results from external forces. After taking into account equilibrium of forces in x direction and Eq. (1) we obtain motion equation expressed in terms of displacement u of the form

$$\ddot{u} - \mu \frac{\partial^2 \ddot{u}}{\partial x^2} - c^2 \frac{\partial^2 u}{\partial x^2} - c^2 \tau_0^{\gamma} D^{\gamma}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{2}$$

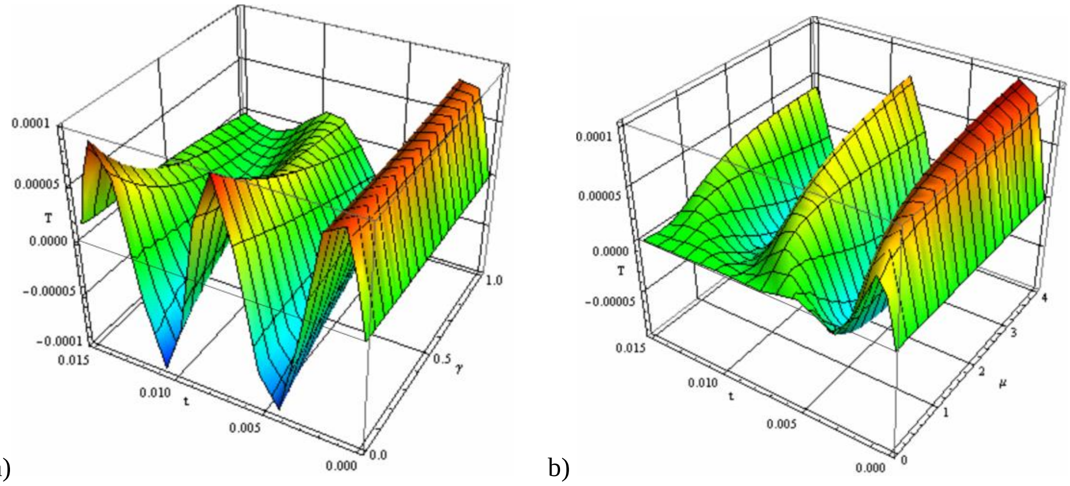
where $c=\sqrt{E_0/\rho}$ is wave propagation velocity and ρ is density of a nanorod. We assume the solution of the Eq. (2) as $u=\sum_{n=1}^{\infty}X_n T_n$ with X_n denoting the amplitude function and T_n denoting the time function. The mode shape solutions for the C-F nanorod are similar to those obtained for a classical C-F rod. However, solution of the differential equation in terms of a time function differs from the solution for a classical rod. The fractional order differential equation for the time function is of the form

$$d\,\ddot{T}_n + f\,D^{\gamma}\left(T_n\right)+\omega_0^2 T_n = 0 \qquad (3)$$

for $d=1+\mu\lambda_n^2$, $f=\omega_0^2\tau_0^{\gamma}$, $\omega_0^2=c^2\lambda_n^2$ and where λ_n, $n=1,2,...,\infty$ are characteristic values which are obtained from characteristic equation and corresponding boundary conditions. To solve Eq. (3) one may use numerical approximation methods or some of the analytical methods available in the literature [3]. Here, we applied Laplace transform method. Obtained solution in the Laplace domain is expanded into the convergent series [4, 5]. Then, using inverse Laplace transform we obtained the solution for the time function in time domain and for the *n*-th mode as follows

$$T_n = T_0\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\binom{k}{j}\frac{m^j v^{(k-j)}t^{2k-\gamma j}}{\Gamma(2\kappa+1-\gamma j)}+\dot{T}_0\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\binom{k}{j}\frac{m^j v^{(k-j)}t^{2k+1-\gamma j}}{\Gamma(2\kappa+2-\gamma j)} \qquad (4)$$

where $m=f/d$, $v=\omega_0^2/d$ and T_0 and $\dot{T}_0$ are initial conditions. For the fractional order initial conditions appearing in the Laplace transform of fractional derivative it is assumed that are equal to zero. On Figs. 1 a) and b). are plotted values of time function in time t for different values of γ and μ. Numerical examples are performed for the following values of parameters: $n=1$, $L=1.1$ [nm], $\tau_0=0.001$ [ns$^{\alpha}$], $E_0=1.1$[TPa], $\rho=2300$ [kg/m^3] and $e_0a=0-2$ [nm].



a) b)

Fig. 1. Values of time function T in time t for a) different values of γ, b) different values of μ.

Conclusions

From presented plots it is obvious that for increase of the order γ of fractional derivative from zero to one we have a smooth transition from harmonic to damped vibration. In addition, it can be revealed that change of the nonlocal parameter μ has significant influence on damped properties of the system i.e. an increase of the parameter decreases the vibration damping properties and influence of the parameter τ_0. However, power series solution is convergent only for smaller times. Therefore, to improve this study other methods for solving the fractional order differential equation need to be considered.

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References

[1] Eringen A. C. (1983) On differential-equations of nonlocal elasticity and solutions of screw dislocations and surface waves. *J Appl Phys* **363**:4703-4710.

[2] Adhikari S., Murmu T., McCarthy M.A. (2013) Dynamic finite element analysis of axially vibrating nonlocal rods. *Finite Elem Anal Des* **63**:42-50.

[3] Mainardi F. (2010) Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, LON.

[4] Hedrih K., Filipovski A. (2002) Longitudinal creep vibrations of a fractional derivative order rheological rod with variable cross section. *Facta Univ Ser Mech Automat Control Robot* **3**:327-349.

[5] Atanacković, T. M., Dolićanin, D. Ć., & Pilipović, S. (2011) Forced oscillations of a single degree of freedom system with fractional dissipation. Scientific Publications of the State University of Novi Pazar Series A: Applied Mathematics, Informatics and mechanics, **3**:1-10.